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Towards a formal taxonomy of hybrid uncertainty representations

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Abstract

In this paper we present some ideas about how to formally relate various uncertainty representations together in a taxonomic structure, capturing both syntactic and semantic generalization. Fuzziness and nonspecificity are presumed as primitive concepts of uncertainty, and transitive and intransitive methods operating with nonspecificity and fuzziness are introduced to generate a base class of hybrid uncertainty representational forms. Additive, maximal, and interval constraints then complete the characterization of the most important hybrid forms.

Keywords: General information theory; Fuzziness; Hybrid uncertainty; Nonspecificity; Dempster–Shafer theory of evidence; Random sets; Possibility theory; Evidence sets; Probability theory

1. Introduction

Recent years have seen a proliferation of methods in addition to probability theory to represent information and uncertainty. These new mathematical structures include fuzzy sets and systems [16], fuzzy measures [25], rough sets [17], random sets [6,13] (Dempster–Shafer bodies of evidence [7,20]), possibility distributions [2], imprecise probabilities [24], etc. We can identify these fields

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collectively as General Information Theory (GIT) [14]. So it is clear that there is a pressing need for the GIT community to synthesize these methods, searching out larger formal frameworks within which to place these various components with respect to each other. And indeed there is a growing movement in that direction [4,15].

There has also been significant work to develop the semantic relations among these components of GIT [21]. Each was originally intended to capture different semantic aspects of uncertainty and information. Traditionally, these semantic criteria include such categories as fuzziness, vagueness, nonspecificity, conflict, imprecision, belief, plausibility, randomness, etc.

Fig. 1, adapted from Klir and Yuan [16], p. 268, shows a typical example of the understood relations among broad classes of uncertainty types, and acts as our point of departure. Klir and Yuan classify uncertainty into two main forms: *ambiguity* and *fuzziness*. Ambiguity is further divided into the categories of *nonspecificity* and *conflict*.

- *Ambiguity:* Webster [26] defines ambiguity as:

1. Doubtfulness or uncertainty of meaning or intention: to speak with ambiguity.
2. The condition of admitting more than one meaning.

Mathematically, ambiguity is identified with the existence of one-to-many relations, that is, when several alternatives exist for the same question or proposition. Nonspecificity is associated with unspecified alternatives, and conflict with the existence of several alternatives with some distinctive characteristics. Possibility theory and probability theory are therefore appropriate formalisms to represent these cases, respectively. Dempster–Shafer (DS) Theory [15,20] provides an ideal framework for the study of ambiguity in general, as it both enlarges the scope of traditional probability theory, and includes possibility theory as a special case.

- *Fuzziness*: Fuzziness is identified with lack of sharp distinctions. Synonyms of "fuzzy" include: *blurred, indistinct, unclear, vague, ill-defined, out of focus, indefinite, shadowy, dim, obscure, misty, hazy, murky, foggy*, and *confused* [26]. Fuzzy logic (or fuzzy set theory) is usually used to formalize this kind of uncertainty. In fuzzy logic the truth value of a proposition ranges between 0 and 1. The amount of fuzziness of a fuzzy logic proposition is defined as the lack of distinction between the proposition and its negation (or between a set and its complement in fuzzy set theory) [27,19].

> Indeed, it is precisely the lack of distinction between sets and their complements that distinguishes fuzzy sets from crisp sets. The less a set differs from its complement, the fuzzier it is [15], p. 298.

In other words, the more something is and is not, *at the same time*, the fuzzier it is. As degree of truth substitutes for binary truth, we find that what is true to a degree, is also false to the inverse degree. In the limit, when something is true and false to the same degree we have paradox.

The authors have been doing significant work with *hybrid* mathematical structures which, it would seem, thereby capture or represent multiple forms of uncertainty. In particular we have been working with random sets [10] and intervals [11,12], which combine nonspecificity and randomness; and with evidence sets [18,19], which introduce fuzzy membership weighted by a probability restriction from the DS theory of evidence [20]. We therefore have an interest in how these hybrid mathematical forms can be incorporated both within a larger mathematical framework, and within the understanding of the interactions among these semantic uncertainty categories.

In this paper we present ideas about how to develop a simple, exhaustive, formal taxonomic basis for the representation of hybrid uncertainty forms sufficient to accommodate these particular structures, and other hybrid and complex uncertainty representation forms. In so doing we depart from the view represented in Fig. 1, proposing fuzziness and nonspecificity as the primitive forms of uncertainty, from which the others are derived by combination and restrictive constraints.

Further, we make the mathematical development of the formalism interactive with their semiotic basis in different semantic interpretations. In this way we aim not only to achieve greater mathematical elegance and generalization, but also a greater semantic coherence among many possible interpretations.

2. The semiotics of uncertainty representations

Our basic assumption, derived from the semiotic perspective on formal language [5], is that mathematical systems are fundamentally independent of their interpretations. That is, we are free to interpret the symbols and productions of a mathematical system in any way we choose, constrained only by the internal logic and consistency of the formal system itself.

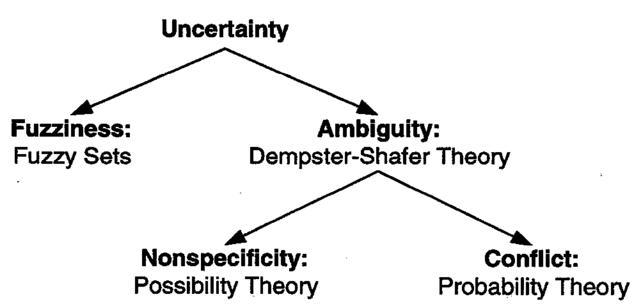


Fig. 1. A semantic taxonomy of uncertainty types (adapted from Klir and Yuan [16]).

Of course, this does not require that formalisms and their interpretations do not interact at all. First, real formalisms are almost all developed within a specific semantic context, which is surely not harmful. Then, although different formalisms, with different axiomatic bases, do not *require* particular interpretations, they do provide different *abilities* to represent natural language concepts or scientific applications.

Furthermore, in general a formalism might have *multiple* possible interpretations, each of which is "valid" for those who find a benefit in using one in that way. Similarly, it might be possible, and even desirable, to represent a certain semantic content in more than one formal system. Ideally, syntactic (mathematical) generalization can both aid and be aided by the semantic analysis available in terms of the conceptual categories outlined above.

Thus we arrive at a picture illustrated in Fig. 2. At the syntactic level, various mathematical systems have formal entailments among them, as indicated by the dashed arrows. Each may also have multiple semantic interpretations, as indicated by the solid arrows, and vice versa. What is demanded is that the mathematical and semantic development go on in the context of each other. So, for example, if a mathematical system has a particular interpretation, and at the same time a formal relation to another mathematical system, then we should attempt to interpret that second system in the original semantic context.

A specific example using the classical structures for uncertainty representation in GIT is shown in Fig. 3. Fuzzy sets are almost always interpreted as linguistic variables modeling the fuzziness of human language, and probability distributions as a constraint on the likelihood or frequency of occurrence. Similarly, possibility distributions are commonly interpreted as forms of "elastic constraint" or graded nondeterminism, and simple intervals as results of imprecise observations or measurements. But formally, we know that both probability and possibility distributions are specialized fuzzy sets, and intervals are specialized possibility distributions. So it is incumbent on us to at least *consider*, for example, interpreting a probability distribution as a linguistic variable, or an additive fuzzy set as a statement of likelihood.

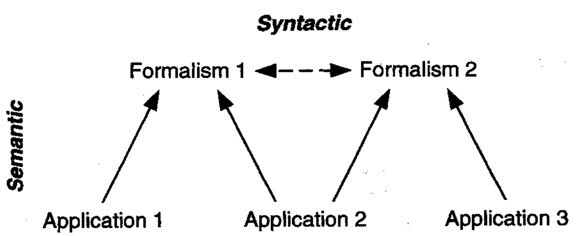


Fig. 2. Formalisms and their applications.

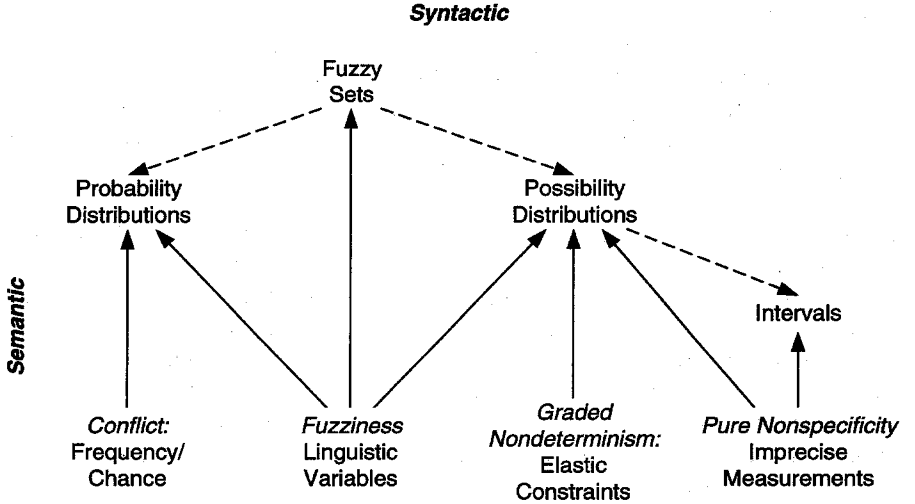


Fig. 3. Some formalisms and their applications in GIT.

3. Mathematical definitions

Assume a universe of discourse $\Omega = \{\omega\}$ of some as yet unspecified cardinality. Sometimes below we restrict $\Omega = \{\omega_i\}, 1 \leqslant i \leqslant n$ to be finite. Other times, $\Omega = \mathbb{R}$ or $\Omega = [0, 1]$.

Denote a *subset* $A \subseteq \Omega$, and a *class* $\mathscr{C} = \{A\} \subseteq \mathscr{P}(\Omega)$, where $\mathscr{P}(\Omega)$ is the power set (set of all subsets) of Ω. A *fuzzy subset* of Ω, denoted $\tilde{A} \tilde{\subseteq} \Omega$, is defined by its membership function $\mu_{\tilde{A}} : \Omega \mapsto [0, 1]$, also denoted as just $\tilde{A}$. The value $\tilde{A}(\omega)$ is usually interpreted as the degree or extent to which $\omega \in \tilde{A}$.

Let an *operator* $\oplus$ be a binary function $\oplus : \mathbb{R}^2 \mapsto \mathbb{R}$ which is commutative, associative, usually monotonic, and with identify 0. For a given fuzzy set $\tilde{A}$, use $\oplus_{\omega \in A} \tilde{A}(\omega)$ as the appropriate operator notation for applying $\oplus$ over the elements of $\tilde{A}$. If $\oplus_{\omega \in \Omega} \tilde{A} = 1$ (or, if Ω is finite, $\oplus_{i=1}^{n} \tilde{A}(\omega_i) = 1$) then $\tilde{A}$ is said to be *normal* by $\oplus$, or to be a *distribution* [10].

When $\oplus = +$, so that $\sum_{\omega \in \Omega} \tilde{A}(\omega) = 1$ (or $\sum_{i=1}^{n} \tilde{A}(\omega_i) = 1$), then $\tilde{A}$ is an additive *probability distribution*. When $\oplus = \vee$, where $\vee$ is the maximum operator, so that $\sup_{\omega \in \Omega} \tilde{A}(\omega) = 1$ ($\vee_{i=1}^{n} \tilde{A}(\omega_i) = 1$), then $\tilde{A}$ is a maximal *possibility distribution*.

Denote the set of all fuzzy subsets of Ω as the fuzzy power set $[0, 1]^{\Omega}$. The weights of a *type-2 fuzzy subset* are themselves fuzzy subsets of $[0, 1]$, and are thus defined by membership functions of the form $\tilde{A} : \Omega \mapsto [0, 1]^{[0,1]}$. A *level-2 fuzzy subset* is a fuzzy subset of the fuzzy power set of Ω, and thus defined by membership functions of the form $\tilde{A} : [0, 1]^{\Omega} \mapsto [0, 1]$.

Given a probability space $\langle X, \Sigma, \Pr \rangle$, where $\langle \cdot \rangle$ denotes a vector, then a function $S : X \mapsto \mathscr{P}(\Omega) - \{\emptyset\}$, where $-$ is set subtraction, is a *random subset* of Ω if S is Pr-measurable, so that $\forall A \subseteq \Omega, S^{-1}(A) \in \Sigma$. Thus a general random set S associates a probability $(\Pr \circ S^{-1})(A)$ to each $A \subseteq \Omega$. Let $m : S(X) \mapsto [0, 1]$, where $S(X) \subseteq \mathscr{P}(\Omega) - \emptyset$ is the image of S, and $m := \Pr \circ S^{-1}$. The *focal set* of a random set S is then defined as $\mathscr{F}(S) := \{A \in S(X)\colon m(A) > 0\}$.

When $\mathscr{F}(S)$ is finite (for example, when either X or Ω is finite), then it is common (e.g. [3]) to much more simply begin by defining $m : \mathscr{P}(\Omega) \mapsto [0, 1]$ as an *evidence function*, with the restrictions that $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$. Then $\mathscr{S} := \{\langle A_j, m_j \rangle\}$ is called a *finite random subset* (or just random set where possible without confusion) of Ω, where $1 \leqslant j \leqslant N := |\mathscr{S}| \leqslant 2^n, m_j := m(A_j)$, and $m_j > 0$. Here the focal set becomes $\mathscr{F}(\mathscr{S}) := \{A_j\colon m_j > 0\}$. Also, for our purposes we can regard a random set $\mathscr{S}$ as a fuzzy subset of $\mathscr{P}(\Omega)$, so that $\mathscr{S} : \mathscr{P}(\Omega) \mapsto [0, 1]$ and $\mathscr{S} = m$, with the additional additivity requirement that $\sum_{A \in \mathscr{P}\Omega} \mathscr{S}(A) = 1$, and such that $A \in \mathscr{F}(\mathscr{S}) \leftrightarrow \mathscr{S}(A) > 0$.

Random set theory is formally equivalent to the DS theory of evidence [7,20]. Here m is called a *basic probability assignment*, and $\{\langle A_j, m_j \rangle\}$ a *body of evidence*. In DS theory emphasis is placed on the belief and plausibility fuzzy measures $\mathrm{Bel}, \mathrm{Pl} : \mathscr{P}(\Omega) \mapsto [0, 1]$, where $\forall A \subseteq \Omega$

$$\mathrm{Bel}(A) := \sum_{A_j \subseteq A} m_j, \qquad \mathrm{Pl}(A) := \sum_{A_j \perp A} m_j$$

and $A \perp B := A \cap B \neq \emptyset$.

Given a class $\mathscr{C} \subseteq \mathscr{P}(\Omega)$, denote $\mathscr{R}(\mathscr{C}) := \{\mathscr{S} : \mathscr{F}(\mathscr{S}) \subseteq \mathscr{C}\}$ as the set of all finite random subsets of Ω whose focal sets are all in $\mathscr{C}$. Note that $\forall \mathscr{S} \in \mathscr{R}(\mathscr{C}), |\mathscr{F}(\mathscr{S})| < \infty$. Denote

$$\mathscr{D} := \{I := [I_l, I_u]\colon I_l, I_u \in \mathbb{R}, I_l \leqslant I_u\},$$
$$\mathscr{D}(I) := \{I' \in \mathscr{D} : I' \subseteq I\}, \quad I \in \mathscr{D}$$

as the class of all closed interval subsets of $\mathbb{R}$, and the class of all closed subintervals of $I \in \mathscr{D}$, respectively. A *random interval* $\mathscr{A}$ is a member of $\mathscr{R}(\mathscr{D})$ [11,12].

Interval-valued fuzzy subsets are defined by membership functions of the form $\tilde{A} : \Omega \mapsto \mathscr{D}([0, 1])$, so that their membership grades are closed interval subsets of $[0, 1]$. Rocha [18,19] has generalized interval-valued fuzzy sets by introducing *evidence sets*, defined by membership functions of the form $\tilde{A} : \Omega \mapsto \mathscr{R}(\mathscr{D}([0, 1]))$. Thus an evidence set maps each element of Ω to a body of evidence whose focal elements are closed intervals of $[0, 1]$, or in other words to a random subinterval of $[0, 1]$.

4. Methodology

We now outline our overall method for developing this taxonomic hierarchy in a number of steps:

1. We begin with both a mathematical and a semantic foundation based on the departure from certainty in two independent directions:
   (a) *Nonspecificity* as the fundamental action of forming a homogeneous collection of multiple possibilities; and
   (b) *Fuzziness* as the fundamental action of allowing a fuzzy degree of truth for a previously certain qualification, thereby creating a heterogeneous collection of a possibility and its weight.
2. We then proceed to provide two methods for the construction of hybrid forms for uncertainty representation based on these primitive *syntactic* actions of collecting and weighting:
   (a) A *transitive* method generates a lattice of possible forms.
   (b) An *intransitive* method generates a tree of possible forms represented as postfix strings.

   These methods end up being largely, but not completely, equivalent, as discussed below.
3. The next step is to apply meaningful constraints on the weightings, including *additive*, *maximal*, and *interval* constraints.
4. The final step is to consider the meaningfullness and usefullness of the remaining structures, and eliminate irrelevant forms.

Now we discuss aspects of this method in more detail, referring ahead sometimes to the results which are shown in Figs. 4–7, and discussed in detail in Section 5.

4.1. Foundations in primitive uncertainty

Consider first that our universe of discourse is finite, with $\Omega = \{a, b, \ldots, z\}$. We then want to describe a situation in which we ask a question of the sort "what is the value of a variable x which takes values in Ω?".

We introduce the assumption here that x can actually only take a unique value in Ω, that is that $x \in \Omega$. In this way we distinguish uncertainty representations as *disjunctive* rather than *conjunctive*: x takes on exactly one value in Ω, and we are interested in characterizing our *knowledge*, and its certainty or uncertainty, as to which value that is.

When there is no uncertainty, or in other words complete certainty, we have a single alternative, say $x = a$. Or, in logical terms, we would say that the proposition p: "the value of x is a", is TRUE.

There are then two primitive operations which can change our knowledge of x, each of which represents a different form of uncertainty.

• *Nonspecificity:* This operation introduces more than one possible answer to the question, while leaving the answers themselves unqualified. This kind of pure nonspecificity implies that our knowledge of which element x is not restricted to a unique a. In other words, we establish a *collection* of possible values. Mathematically, this operation introduces a subset of possible answers. Logically, we now have a set P of propositions p, one for each member of some group of the $\omega \in \Omega$, but the value of each proposition is TRUE. Formally, this operation is represented as a simple *homogeneous* collection, a subset $A \subseteq \Omega$, for example $A = \{a, b\} \in \mathscr{P}(\Omega)$.

• *Fuzziness:* This operation introduces a fuzzy degree of truth to qualify the answer to the question, while still leaving only a *single* answer. This kind of pure fuzziness implies that our knowledge of the value of x is fuzzy or blurred, as it rests on just a single value a, but to a degree. Mathematically, a numerical weighting $\mu(a) \in [0, 1]$ is introduced on the answer a. Logically, the proposition p is still "the value of x is a", but it's truth value is $\mu(p) \in [0, 1]$. Formally, this operation is represented by a *heterogeneous* collection, a vector $\langle x, \mu(x) \rangle \in \Omega \times [0, 1]$, for example $\langle a, 0.3 \rangle$.

We assert not only that collections and weighted elements are the paradigmatic forms of primitive uncertainty, but also that the *actions* of collecting or weighting alternatives with a fuzzy degree are a sufficient *procedural* foundation for the production of more complex forms of uncertainty structures. Indeed, these two actions are used below as the two basic operations of an *uncertainty generation grammar* leading to the construction of such mathematical structures.

We also assert that this is well justified and motivated both in its mathematical simplicity and its semantic coherence. In other words, we believe that all other more complex forms entail fuzziness, nonspecificity, or both, at both the mathematical and semantic levels, although perhaps with the inclusion of additional constraints, which we will introduce below.

One consequence of this view is to reject the idea that probabilistic conflict or possibilistic imprecision are independent categories of uncertainty. Rather, each is a more complex expression of uncertainty involving each of these fundamental forms of fuzziness and nonspecificity.

In other words, a probability distribution in general does not represent the semantic category of conflict in any *pure* sense. Rather, it also entails nonspecificity, in the collection of elements of the universe on which it takes values; and fuzziness, in the various weighting that those values can take, albeit they have an additive constraint present among them.

4.2. Taxonomic development

In the taxonomic development itself, the intention is to start from the state of pure certainty ($x = a$), introduce the first principles of primitive uncertainty representations and transformations (collecting and weighting), and apply them iteratively to generate the basic framework of all the more complex forms.

We have identified two distinct overall methods.

- *Transitive:* In this approach we consider each entity as either a homogeneous collection (a subset), or a heterogeneous collection (a vector). We then alternatively apply fuzzy weighting and collecting to either the whole or the parts of the collection. For example, weighting the parts of a simple collection, yielding a classical fuzzy set, suggests weighting the parts of a weighted element separately. In this way we end up applying transformations of weighting and collecting consistently to all entities seen. Transitivities exist in the sense of multiple paths of transformations resulting in the same structures. Transformations overall are irreversible, and a lattice structure results. The transitivities in the lattice define equivalence classes of transformations. Some of these equivalent transformations involve transformations through intermediate forms. For example, weighting the parts of a subset is equivalent to collecting fuzzy elements, each resulting in a fuzzy set. Or, making a whole collection a weight is equivalent to making the weights of a weighted element collections. These paths and the intermediate entities are kept distinct. Other equivalent transformations do not involve intermediate entities. For example, making the parts of a collection is equivalent to combining separate collections into a new collection. These collection transformations are identified in the diagram.
- *Intransitive:* In this approach generalizations apply in a strict sequence, so that a generalization is only applied to the portion of its representation which was last generalized. Each of these steps can be seen as a production rule. This allows the transformations to be reversible, and introduces a significant amount of order to the space of hybrid representations. An uncertainty structure is uniquely defined by the sequence of rules that produce it from the initial no uncertainty situation ($x = a$). Furthermore, this history of generalization constraint the type of uncertainty generalization that can be applied to the present structure, since only the portion of its representation that was previously generalized can be generalized again with one of the two production rules. What results is a tree structure generated by a simple postfix string production system. For the grammar, the atoms are $\{F, N\}$ for fuzziness and nonspecificity, and the productions are:

$$X \mapsto XF, \quad X \mapsto XN.$$

So using just these two concepts of fuzziness and nonspecificity, with either method we can describe a variety of foundational representations, including a variety of simple sets, weighted elements, and classical fuzzy sets. After more iterations, with other levels of fuzziness or nonspecificity added in, more complex forms such as type 2 fuzzy sets, set-valued fuzzy sets, "fuzzy classes", and "fuzzified classes", appear.

In both methods, each application of a transformation introduces another "level" of either nonspecificity or fuzziness to the prior structure. We use the notations $|F| \in \mathscr{W} := \{0, 1, 2, \ldots\}$ and $|N| \in \mathscr{W}$ to indicate the number of levels of fuzziness and nonspecificity present, respectively. Denote $|NF| := \langle |N|, |F| \rangle \in \mathscr{W}^2$, which is then the overall "uncertainty cardinality" of a given structure, however it was generated.

4.3. Syntactic constraints

Once the overall taxonomic structure has been generated, it is possible to identify structures which have both fuzziness and nonspecificity present, that is where $|NF| \geqslant \langle 1, 1 \rangle$. Under these conditions, additional constraints can be applied to achieve traditional GIT structures.

Normalization: Fuzzy sets are one of the simplest structures which contain both fuzziness and nonspecificity, in that they are a collection of weighted elements. If a normalization operator $\oplus$ is applied to this collection of weights, the resulting structures become appropriately distributed. We identify two of particular significance:

- *Additive:* When $\oplus = +$, then the resulting structure is probabilistically distributed. For example, fuzzy sets become probability distributions, and fuzzified classes become random sets (see below).
- *Maximal:* When $\oplus = V$, then the resulting structure is possibilistically distributed. For example, fuzzy sets become possibility distributions, and fuzzified classes become possibilistic sets.

Universe of discourse: Constraints on the base universe of discourse Ω are also available and used somewhat in the results below. For example, we can let $\Omega = \mathbb{R}$ or $\Omega = [0, 1]$.

Intervals: What we call "set-valued fuzzy elements", or structures with the general form $\langle \omega_i, M(\omega_i) \rangle$, where $M(\omega_i) \in \mathscr{P}([0, 1])$, are generated very early in either the transitive or intransitive taxonomies. These are the simplest structures which contain both fuzziness and nonspecificity in the form of a weight which, conversely to fuzzy sets, is *itself* a collection. Such structures, and others closely related to them, for example set-valued fuzzy sets, can be constrained to make $M(\omega_i) \in \mathscr{D}([0, 1])$, so that weights are interval subsets of $[0, 1]$. This results, for example, in interval-valued fuzzy elements and interval-valued fuzzy sets.

Given an existing set-valued structure, the method to affect the transformation to an interval-valued structure is actually quite simple. Given $M(\omega_i) \in \mathscr{P}([0, 1])$, where $|M(\omega_i)| < \infty$, construct a new weight $M'(\omega_i) \in \mathscr{D}([0, 1])$ as the simple closure

$$M'(\omega_i) := [\min\{x \in M(\omega_i)\}, \max\{x \in M(\omega_i)\}].$$

These additional mathematical constraints allow greater semantic expressibility. Specifically, the additivity of probability allows the expression of conflict, or randomness. And the maxitivity of possibility allows the expression of ordinal concepts surrounding distances and capacities [9].

4.4. Semantic interpretation

Finally, now that these basic structures have been identified, they are available for interpretation. Attempting to semantically identify these various structures results in three possible situations:

- Traditional forms are recovered, for example fuzzy or random sets.
- Nontraditional, but meaningful and suggestive, forms are generated. This is the case, for example, with the structure we call the fuzzy class, a collection of arbitrarily weighted subsets, whose additively constrained form is a random set.

Both of these first two forms are identified on the diagrams with an appropriate label. But in addition a final case can result:

- Nontraditional, and likely unmeaningful forms are generated.

These are labeled in the diagrams using quotation marks.

5. Results

We now describe some results of our approach. First we discuss the specific application of both the transitive and intransitive methods. Then the particular basic and constrained forms are enumerated.

5.1. Methods

Results for the transitive method are shown in Figs. 4 and 5, and those for the intransitive method are shown in Figs. 6 and 7.

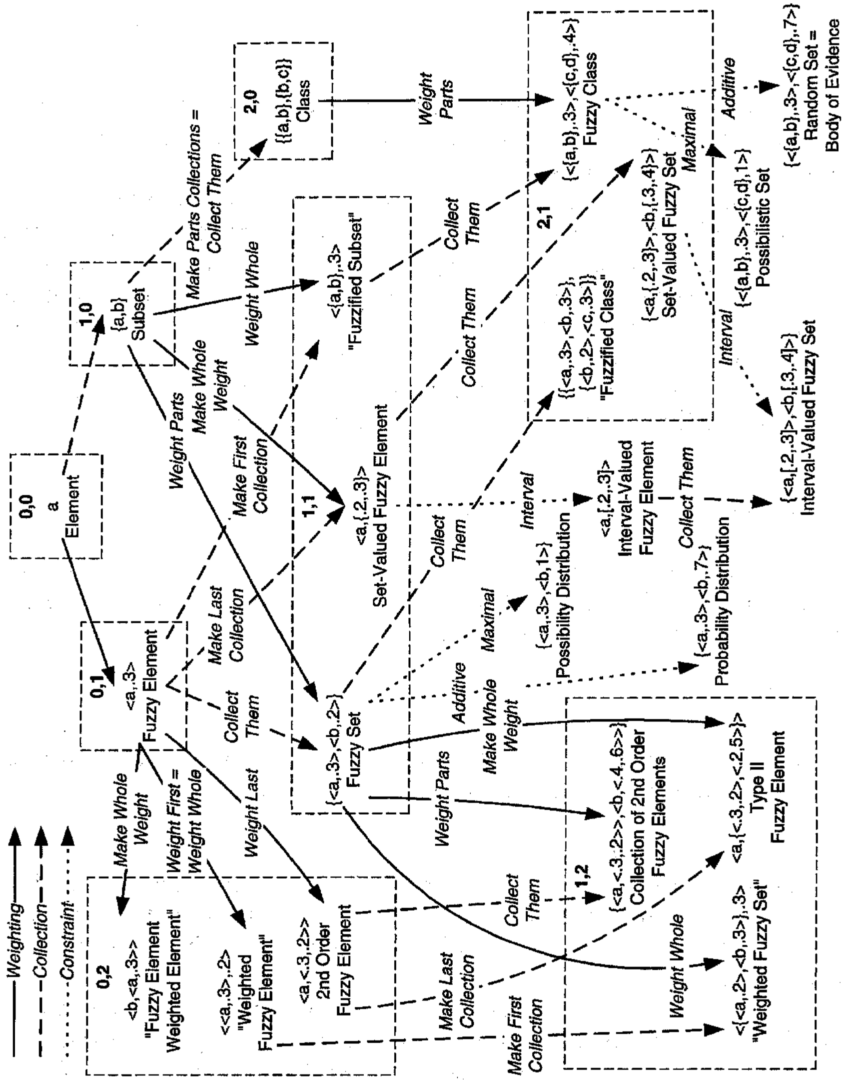
5.1.1. Transitive method

Partial results for the transitive method are shown in Figs. 4 and 5. Fig. 4 shows the base forms through $|N| + |F| \leqslant 3$, and some of the resulting constrained forms. Fig. 5 continues with some structures with $|N| + |F| \geqslant 3$, concluding with evidence sets.

In the figures, each particular form is identified by its label and canonical example. Dashed boxes surround forms with identical $|NF|$. Transformations operate on parts or whole, and are so identified. Not all possible transformations are shown, in order to make the complexity of the diagrams reasonable to handle.

Transformations are of three distinct kinds:

1. *Weighting:* Solid arrows indicate fuzzy weighting. Weighting transformations can be one of the following:


Weighting
Collection
Constraint
0,0
a
Element
1,0
{a,b}
Subset
2,0
{{a,b},{b,c}}
Class
Make Parts Collections =
Collect Them
Weight Parts
Weight Whole
<{a,b},.3>
"Fuzzified Subset"
Make Whole
Weight
Weight Parts
Collect Them
2,1
{<{a,b},.3>,<{c,d},.4>}
Fuzzy Class
Additive
{<{a,b},.3>,<{c,d},.7>}
Random Set =
Body of Evidence
Maximal
{<{a,b},.3>,<{c,d},1>}
Possibilistic Set
{<a,[.2,.3]>,<b,[.3,.4]>}
Set-Valued Fuzzy Set
{{<a,.3>,<b,.3>},
{<b,.2>,<c,.3>}}
"Fuzzified Class"
Interval
{<a,[.2,.3]>,<b,[.3,.4]>}
Interval-Valued Fuzzy Set
Collect Them
Make First
Collection
1,1
<a,{.2,.3}>
Set-Valued Fuzzy Element
Interval
<a,[.2,.3]>
Interval-Valued
Fuzzy Element
Collect Them
0,1
<a,.3>
Fuzzy Element
Make Last
Collection
Collect
Them
{<a,.3>,<b,1>}
Possibility Distribution
Maximal
{<a,.3>,<b,.7>}
Probability Distribution
Additive
{<a,.3>,<b,.2>}
Fuzzy Set
Make Whole
Weight
Weight Parts
Make Whole
Weight
Weight First =
Weight Whole
Weight Last
0,2
<b,<a,.3>>
"Fuzzy Element
Weighted Element"
<<a,.3>,.2>
"Weighted
Fuzzy Element"
<a,<.3,.2>>
2nd Order
Fuzzy Element
Collect
Them
1,2
{<a,<.3,.2>>,<b,<.4,.6>>}
Collection of 2nd Order
Fuzzy Elements
<a,{<.3,.2>,<.2,.5>}>
Type II
Fuzzy Element
Weight Whole
<{<a,.2>,<b,.3>},.3>
"Weighted Fuzzy Set"
Make Last
Collection
Make First
Collection

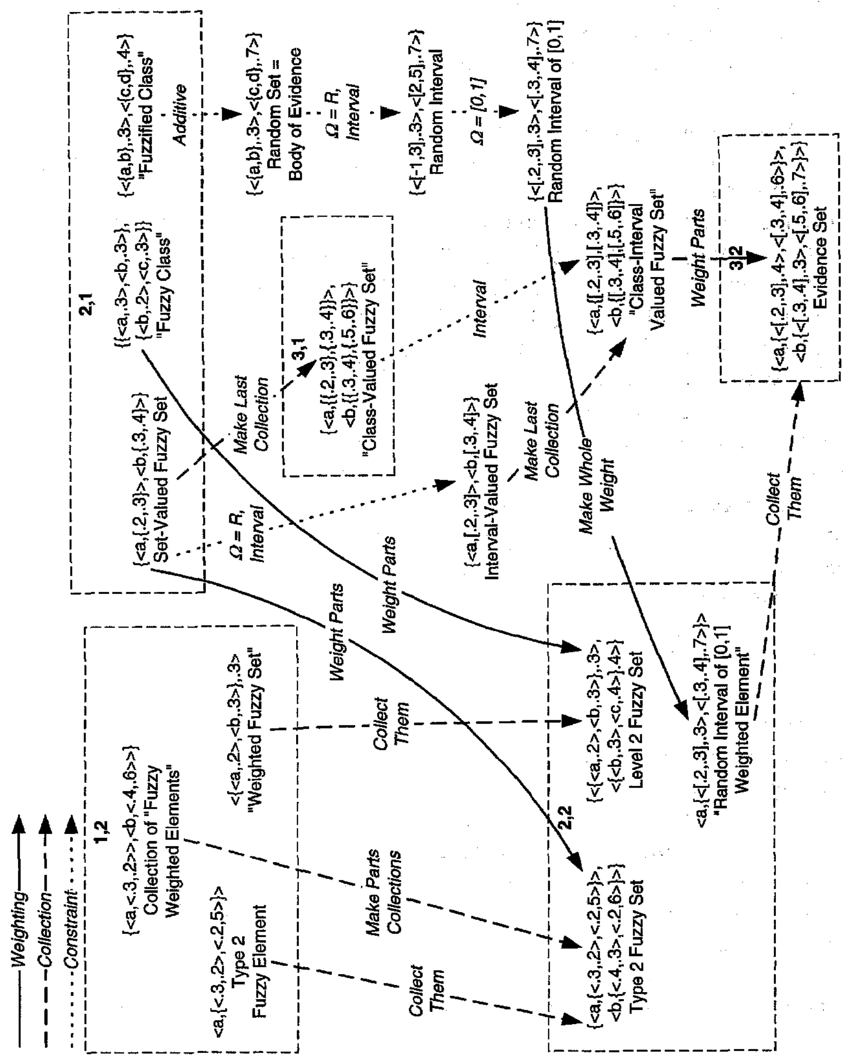


Fig. 5. Partial lattice of uncertainty representations, transitive method, $|N| + |F| \geqslant 3$.

(a) To give weights to any of the *parts* of the preceding structure;
(b) To give a weight to the *whole* of the preceding structure; or
(c) To *make* the whole preceding structure *itself* a weight of a new element.

2. *Collecting:* Dashed arrows indicate nonspecific collecting. Collecting transformations can be either:
   (a) To make any of the *parts* of the preceding structure into a collection; or
   (b) To collect together *wholes* of the preceding structure.
3. *Constraint:* Dotted arrows indicate forms of constraint, and are appropriately labeled.

Note that the basic transformations of collecting and weighting create higher levels of homogeneous and heterogeneous structures, respectively. Thus the base forms are essentially multiply-layered hierarchical structures of alternating homogeneous subsets and heterogeneous vectors.

A note must also be made about the "make whole weight" transformation number 1c above. This method proceeds by taking a given object, for example a subset $\{a, b\} \subseteq \Omega$, and using the *structure* of that object to be the *structure* of a weighting on a new element. Thus this transformation implies that the preceeding structure must be *capable* of being a weight, and thus implies the universe of discourse constraint such that $\Omega = [0, 1]$.

Thus, for example, the transformation from Fig. 4 taking a subset to a set-valued fuzzy element $\{a, b\} \mapsto \langle a, \{0.2, 0.3\}\rangle$, should actually be seen as the series of transformations

$$\{a, b\} \mapsto \{0.2, 0.3\} \mapsto \langle a, \{0.2, 0.3\}\rangle.$$

5.1.2. Intransitive method

Partial results for the intransitive method are shown in Figs. 6 and 7. As above, Fig. 6 shows the base forms through $|N| + |F| \leqslant 3$, and some of the resulting constrained forms, and Fig. 7 continues with some structures with $|N| + |F| \geqslant 3$, concluding with evidence sets.

In these figures, the arrows are similar to those in Section 5.1.1, except that types of fuzziness and nonspecificity transformations are not distinguished. Recall that unlike the transitive method which can use its operations in any part of the representation of an uncertainty structure, the intransitive method is constrained to applying its methods to only the portion which was last generalized.

For example, fuzzy sets are constructed (uniquely) by applying the fuzziness production rule to a crisp set (itself a nonspecific structure). This operator associates a weight with each element of the crisp set. Thus, to generalize fuzzy sets with this method, we can apply either of the weighting or collecting production rules to these weights.

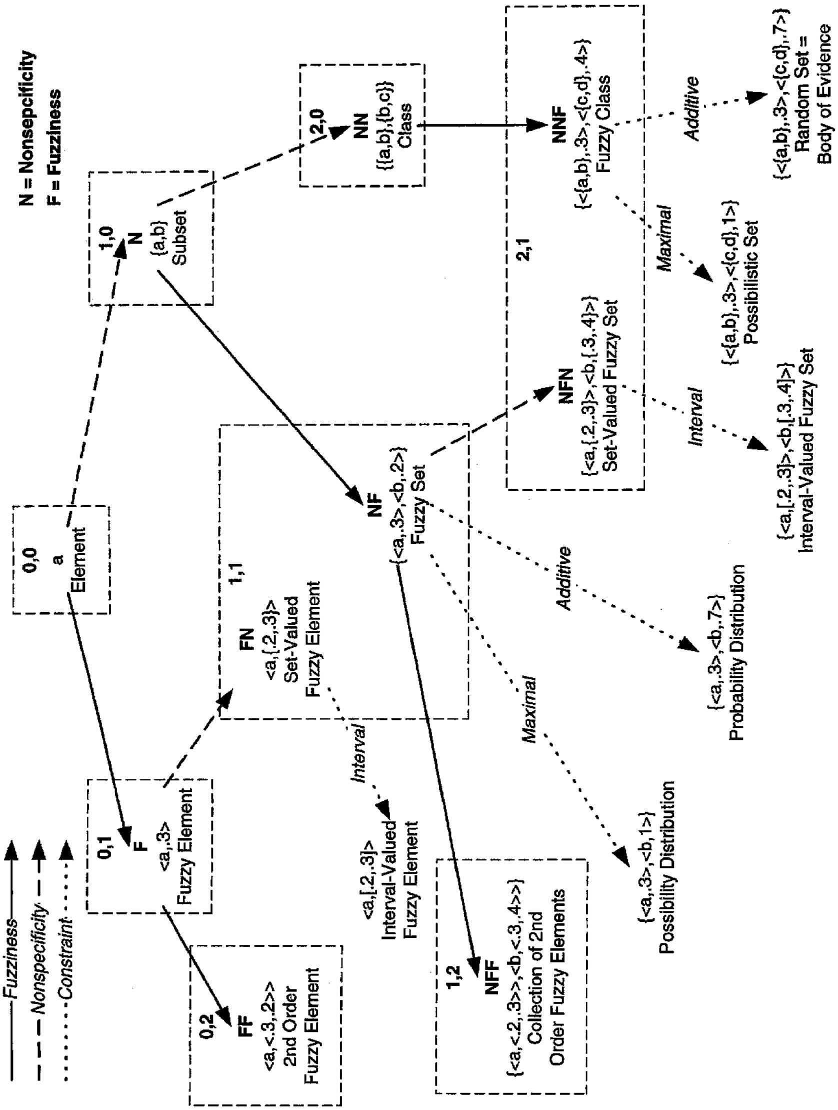


Fig. 6. Partial tree of uncertainty representations, intransitive method, $|N| + |F| \leqslant 3$.

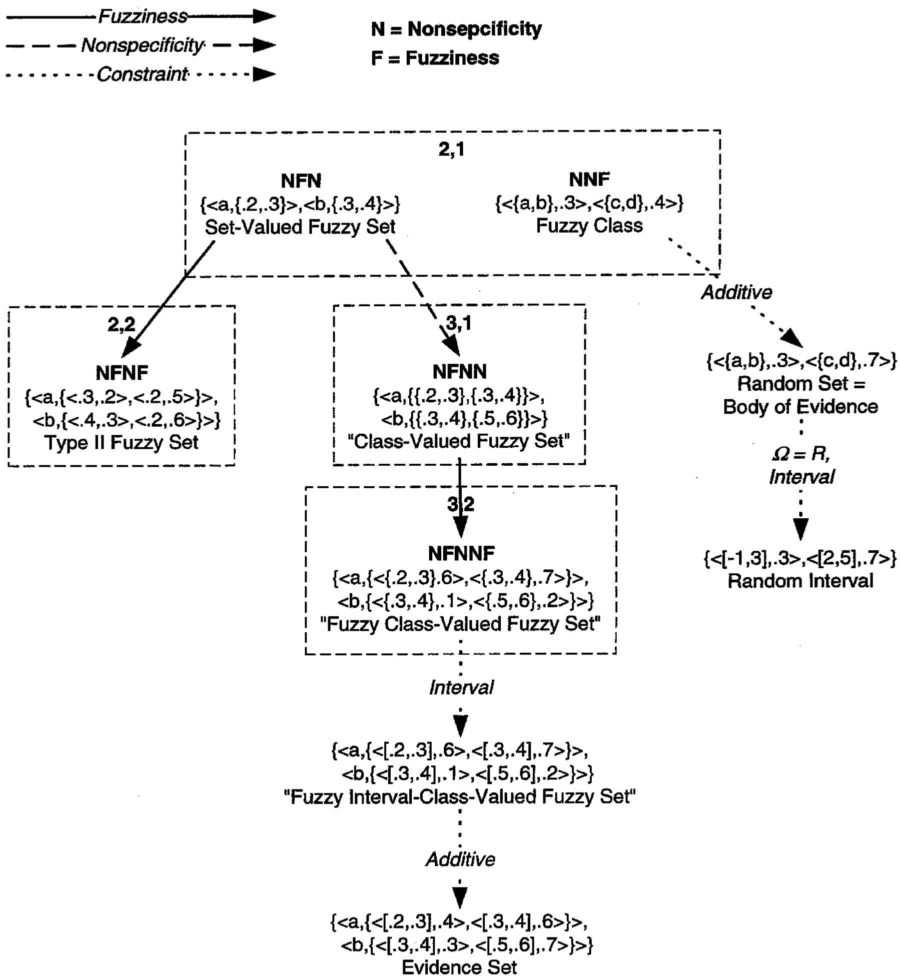


Fig. 7. Partial tree of uncertainty representations, intransitive method, $|N| + |F| \geqslant 3$.

In other words, whatever generalization we pursue, it will always result in some sort of set structure with more and more complicated kinds of weights. In a sense, the sequence of transformation preserves or inherits the primordial structure with $|NF| = \langle 0, 0\rangle$.

5.2. Resulting structures

We now discuss in detail the hybrid information structures generated by these methods.

5.2.1. Basic structures

Table 1 lists the basic structures (those involving uncertainty and fuzziness in various combinations, with no other constraints) generated. The left column indicates $|NF|$. For example, a Type 2 Fuzzy Element has two levels of fuzziness and one level of nonspecificity. For each structure generated, we also indicate a simple canonical example drawn from the universe of discourse $\Omega = \{a, b, c\}$. Finally, we indicate the appropriate label or description of the structure. Some of these labels are novel. Those labels in quotation marks are of questionable semantic significance.

We now describe these forms in detail. They are listed in $|NF|$ partial order, and include an example for each.

$\langle 0, 1\rangle$: *The fuzzy element*, for example $\langle a, 0.3\rangle$, is the single form representing pure fuzzy weighting. It is a single element given a degree of truth. This form is generally not recognized in set theory, but can be considered a fuzzy proposition in fuzzy logic.

$\langle 1, 0\rangle$: The *subset*, for example $A = \{a, b\}$, is the single form of representing pure nonspecific collecting. This has obviously been a standard mathematical form for centuries.

$\langle 0, 2\rangle$: These forms have two levels of fuzziness only.

- *Second order fuzzy element* $\langle a, \langle 0.3, 0.3\rangle\rangle$: In this form, the weight of a fuzzy element is itself given a weight. Although this form is not recognized in itself, it is used in further structures, and forms one of the bases for Type 2 Fuzzy Sets.
- *Weighted fuzzy element* $\langle\langle a, 0.3\rangle, 0.3\rangle$: In this form, a fuzzy element is weighted. It is also not recognized in and of itself, but is used in some later forms.
- *Fuzzy element weighted element* $\langle a, \langle b, 0.3\rangle\rangle$: In this form, a fuzzy element is itself used *as* a weight of an element. Another nonstandard form.

$\langle 2, 0\rangle$: The *class*, for example $\mathscr{C} = \{\{a, b\}, \{b, c\}\}$, is the single form with two levels of nonspecificity only.

$\langle 1, 1\rangle$: These forms have a single level each of fuzziness and nonspecificity, and are thus central in the taxonomic hierarchy. While they include some of the most standard forms in GIT, there are also some novel structures here.

- *Fuzzy set* $\{\langle a, 0.3\rangle, \langle b, 0.2\rangle\}$: Of course, this is the classical form for GIT. In the transitive method, it is derived in a number of ways, and then used to derive many further forms. In the intransitive method, it is derived by collecting fuzzy elements.
- *Set-valued fuzzy element* $\langle a, \{0.2, 0.3\}\rangle$: This is an element whose weight is a collection. Although this is a novel form, it is central to both the transitive and intransitive methods.
- *Fuzzified subset* $\langle\{a, b\}, 0.3\rangle$: This is a weighted subset. Although not recognized on its own, it is used in later forms in both methods.

Table 1
Generated unconstrained hybrid uncertainty forms

$\|NF\|$	Label example
0, 1	Fuzzy element $\langle a, 0.3\rangle$
1, 0	Subset $\{a, b\}$
0, 2	Second order fuzzy element $\langle a, \langle 0.3, 0.3\rangle\rangle$ "Weighted fuzzy element" $\langle\langle a, 0.3\rangle, 0.3\rangle$ "Fuzzy element weighted element" $\langle a, \langle b, 0.3\rangle\rangle$
2, 0	Class $\{\{a, b\}, \{b, c\}\}$
1, 1	Fuzzy set $\{\langle a, 0.3\rangle, \langle b, 0.2\rangle\}$ Set-valued fuzzy element $\langle a, \{0.2, 0.3\}\rangle$ "Fuzzified subset" $\langle\{a, b\}, 0.3\rangle$
1, 2	Type 2 fuzzy element $\langle a, \{\langle 0.2, 0.3\rangle, \langle 0.2, 0.5\rangle\}\rangle$ Collection of second order fuzzy elements $\{\langle a, \langle 0.2, 0.3\rangle\rangle, \langle b, \langle 0.2, 0.5\rangle\rangle\}$ "Weighted fuzzy set" $\langle\{\langle a, 0.2\rangle, \langle b, 0.3\rangle\}, 0.3\rangle$
2, 1	Set-valued fuzzy set $\{\langle a, \{0.2, 0.3\}\rangle, \langle b, \{0.3, 0.4\}\rangle\}$ Fuzzy class $\{\langle\{a, b\}, 0.3\rangle, \langle\{b, c\}, 0.4\rangle\}$ "Fuzzified class" $\{\{\langle a, 0.3\rangle, \langle b, 0.2\rangle\}, \{\langle b, 0.2\rangle, \langle c, 0.3\rangle\}\}$
2, 2	Type 2 fuzzy set $\{\langle a, \{\langle 0.3, 0.2\rangle, \langle 0.2, 0.5\rangle\}\rangle, \langle b, \{\langle 0.4, 0.2\rangle, \langle 0.2, 0.6\rangle\}\rangle\}$ Level 2 fuzzy set $\{\langle\{\langle a, 0.2\rangle, \langle b, 0.3\rangle\}, 0.3\rangle, \langle\{\langle b, 0.3\rangle, \langle c, 0.4\rangle\}, 0.4\rangle\}$
3, 1	"Class-valued fuzzy set" $\{\langle a, \{\{0.2, 0.3\}, \{0.3, 0.4\}\}\rangle, \langle b, \{\{0.3, 0.4\}, \{0.5, 0.6\}\}\rangle\}$
3, 2	"Fuzzy class-valued fuzzy set" $\{\langle a, \{\langle\{0.2, 0.3\}, 0.6\rangle, \langle\{0.3, 0.4\}, 0.7\rangle\}\rangle, \langle b, \{\langle\{0.3, 0.4\}, 1\rangle, \langle\{0.5, 0.6\}, 0.2\rangle\}\rangle\}$

$\langle 1, 2\rangle$: We now consider the first more complex forms, here with two levels of fuzziness and one of nonspecificity.

- *Type 2 fuzzy element* $\langle a, \{\langle 0.2, 0.3\rangle, \langle 0.2, 0.5\rangle\}\rangle$: An element weighted by a fuzzy subset of $[0, 1]$, or weighted by a collection of fuzzy weighted elements. These are one basis of type 2 fuzzy sets.

- *Collection of second order fuzzy elements* $\{\langle a, \langle 0.2, 0.3\rangle\rangle, \langle b, \langle 0.2, 0.5\rangle\rangle\}$: Another basis for type 2 fuzzy sets, these are precisely a collection of second order fuzzy elements.
- *Weighted fuzzy set* $\langle\{\langle a, 0.2\rangle, \langle b, 0.3\rangle\}, 0.3\rangle$ This is a fuzzy set which is then itself given a weight. This is one basis for level 2 fuzzy sets.

$\langle 2, 1\rangle$: These are forms conversely with one level of fuzziness and two of nonspecificity.

- *Set-valued fuzzy set* $\{\langle a, \{0.2, 0.3\}\rangle, \langle b, \{0.3, 0.4\}\rangle\}$: This is a fuzzy set whose weights are collections.
- *Fuzzy class* $\{\langle\{a, b\}, 0.3\rangle, \langle\{b, c\}, 0.4\rangle\}$: A collection of weighted subsets, and the basis for random sets. These have been introduced in the context of fuzzy multi-valued mappings [22,23].
- *Fuzzified class* $\{\{\langle a, 0.3\rangle, \langle b, 0.2\rangle\}, \{\langle b, 0.2\rangle, \langle c, 0.3\rangle\}\}$: What we are calling a fuzzified class is then, conversely, a subset of the fuzzy power set, or just a collection of fuzzy sets. It is another basis for level 2 fuzzy sets in the transitive method.

$\langle 2, 2\rangle$: These forms have both two levels of nonspecificity and two levels of fuzziness. Each is a standard form in fuzzy systems.

- *Type 2 fuzzy set* $\{\langle a, \{\langle 0.3, 0.2\rangle, \langle 0.2, 0.5\rangle\}\rangle, \langle b, \{\langle 0.4, 0.2\rangle, \langle 0.2, 0.6\rangle\}\rangle\}$
- *Level 2 fuzzy set* $\{\langle\{\langle a, 0.2\rangle, \langle b, 0.3\rangle\}, 0.3\rangle, \langle\{\langle b, 0.3\rangle, \langle c, 0.4\rangle\}, 0.4\rangle\}$

$\langle 3, 1\rangle$: We have identified a novel form with three levels of nonspecificity and one of fuzziness. It is available from both the transitive and intransitive methods, and we call it a *class-valued fuzzy set*. Its canonical example is:

$$\{\langle a, \{\{0.2, 0.3\}, \{0.3, 0.4\}\}\rangle, \langle b, \{\{0.3, 0.4\}, \{0.5, 0.6\}\}\rangle\}.$$

Essentially, this results when the weights of a set-valued fuzzy set are made into collections, and is an important step on the road to evidence sets.

$\langle 3, 2\rangle$: When, in the transitive method, a fuzziness transformation is applied to a class-valued fuzzy set in order to give either of the sub-elements of the set-valued fuzzy set a weight, we call the resulting structure a *fuzzy class-valued fuzzy set*, with canonical example:

$$\{\langle a, \{\langle\{0.2, 0.3\}, 0.6\rangle, \langle\{0.3, 0.4\}, 0.7\rangle\}\rangle, \langle b, \{\langle\{0.3, 0.4\}, 1\rangle, \langle\{0.5, 0.6\}, 0.2\rangle\}\rangle\}.$$

This form is essentially an evidence set with none of the necessary additive or interval constraints present.

5.2.2. Constrained structures

Table 2 lists some of the forms which result when syntactic constraints as described in Section 4.3 are applied to base forms. The base forms from which they are derived, and the form of constraint employed, are also listed.

We now describe these in detail:

Probability distribution: Results when an additive constraint is applied to the weights of a fuzzy set.

Possibility distribution: Results when a maximal constraint is applied to the weights of a fuzzy set.

Interval-valued fuzzy element: Results when an interval constraint is applied to the weight (itself a collection) of a set-valued fuzzy element.

Interval-valued fuzzy set: Results when an interval constraint is applied to the weights (themselves collections) of a set-valued fuzzy set. Alternately, it is a collection of interval-valued fuzzy elements.

Random set: Equivalent to a DS body of evidence, results from the application of an additive constraint to the weights of a fuzzy class.

Possibilistic set: This is a relatively new structure, although it has been recognized [22,23]. This is essentially a possibilistic version of a random set, or a class where the elements are weighted in a possibilistically normal manner.

Random interval: Results from the application of an interval constraint on the universe of discourse, or domain, of a random set.

Random interval of $[0, 1]$: Results from further constraining the initial universe of discourse of a random set to be $[0, 1]$.

Random interval of $[0, 1]$ *weighted element:* Results from letting a random interval of $[0, 1]$ weight an element. Example:

Table 2
Constrained hybrid uncertainty forms (FS = Fuzzy Set)

Resulting form	Base form Example	Constraint
Probability distribution	FS $\{\langle a, 0.3\rangle, \langle b, 0.7\rangle\}$	Additive
Possibility distribution	FS $\{\langle a, 0.3\rangle, \langle b, 1\rangle\}$	Maximal
Interval-valued fuzzy element	Set-valued fuzzy element $\langle a, [0.2, 0.3]\rangle$	Interval
Interval-valued FS	Set-valued FS $\{\langle a, [0.2, 0.3]\rangle, \langle b, [0.3, 0.4]\rangle\}$	Interval
Random set	Fuzzy class $\{\langle\{a, b\}, 0.3\rangle, \langle\{b, c\}, 0.7\rangle\}$	Additive
Possibilistic set	Fuzzy class $\{\langle\{a, b\}, 0.3\rangle, \langle\{b, c\}, 1\rangle\}$	Maximal
Random interval	Random set $\{\langle[-1, 3], 0.3\rangle, \langle[2, 5], 0.7\rangle\}$	$\Omega = \mathbb{R}$, Interval
Random interval of [0, 1]	Random interval $\{\langle[0.2, 0.3], 0.3\rangle, \langle[0.2, 0.5], 0.7\rangle\}$	$\Omega = [0, 1]$
"Class-interval valued FS"	Class-valued FS $\{\langle a, \{[0.2, 0.3], [0.3, 0.4]\}\rangle, \{\langle b, \{[0.3, 0.4], [0.5, 0.6]\}\rangle\}$	Interval
"Fuzzy interval-class-valued FS"	Fuzzy class-valued FS $\{\langle a, \{\langle[0.2, 0.3], 0.6\rangle, \langle[0.3, 0.4], 0.7\rangle\},\rangle, \langle b, \{\langle[0.3, 0.4], .1\rangle, \langle[0.5, 0.6], 0.2\rangle\}\rangle\}$	Interval
Evidence set	Fuzzy interval-class-valued FS $\{\langle a, \{\langle[0.2, 0.3], 0.4\rangle, \langle[0.3, 0.4], 0.6\rangle\},\rangle, \langle b, \{\langle[0.3, 0.4], 0.3\rangle, \langle[0.5, 0.6], 0.7\rangle\}\rangle\}$	Additive

$$\langle a, \{\langle [0.2, 0.3], 0.3\rangle, \langle [0.3, 0.4], 0.7\rangle\}\rangle.$$

Class-interval valued fuzzy set: Results from applying an interval constraint to a class-valued fuzzy set, thus making the weights on the elements be collections of intervals. Example:

$$\{\langle a, \{[0.2, 0.3], [0.3, 0.4]\}\rangle, \{\langle b, \{[0.3, 0.4], [0.5, 0.6]\}\rangle\}.$$

Fuzzy interval-class-valued fuzzy set: To be parsed as: (Fuzzy (Interval-Class-Valued)) Fuzzy Set. An interval constraint is applied to the weights of a fuzzy class-valued fuzzy set. In other words, the weights of a fuzzy set are themselves a collection of weighted intervals of $[0, 1]$. Example:

$$\{\langle a, \{\langle [0.2, 0.3], 0.6\rangle, \langle [0.3, 0.4], 0.7\rangle\},\rangle, \langle b, \{\langle [0.3, 0.4], 0.1\rangle, \langle [0.5, 0.6], 0.2\rangle\}\rangle\}.$$

Evidence set: Finally we arrive at our most complex structure, an evidence set with $|NF| = \langle 3, 2\rangle$. It is constructed either by placing an additive constraint on the weights of the intervals of a fuzzy interval-class-valued fuzzy set, or by making a collection of random interval of $[0, 1]$ weighted elements. Essentially, an evidence set is a fuzzy set whose weights are DS bodies of evidence, or random intervals, on $[0, 1]$. Example:

$$\{\langle a, \{\langle [0.2, 0.3], 0.4\rangle, \langle [0.3, 0.4], 0.6\rangle\},\rangle, \langle b, \{\langle [0.3, 0.4], 0.3\rangle, \langle [0.5, 0.6], 0.7\rangle\}\rangle\}.$$

5.3. Methodological comparison

The transitive method can expand fuzzy sets in more ways, for instance by weighting the whole structure and then collecting them to obtain a Level 2 fuzzy set, a structure that is not obtainable with the transitive method.

The intransitive method has the advantage of being reversible, thus defining structures with a unique transformation history. It allows the description of most known, semantically defined, truth-value representations and set structures, including evidence sets.

However, the intransitive method is thereby exploring only subsections of the entire universe of hybrid uncertainty structures which it attempts to simplify. All structures reached by this method are either an extension of a set or of a truth-value. If we start by generalizing the certain situation with the fuzziness transformation, all subsequent uncertainty structures can only be some form of singleton with extended truth-values. If we start with the non-specificity transformation all structures reached are either a crisp collection of singletons with extended truth-values (e.g. fuzzy sets and its descendants), or nested crisp classes whose elements can be ascribed extended truth-value representations.

6. Conclusions and further work

In this paper we have attempted to outline the basis for the development of a formal structure for the generation of structures to represent hybrid forms of uncertainty from a valid semiotic basis. The results presented so far are not complete. For example, the full space of all of the transitive transformations through, say, $|N| + |F| \leqslant 4$ has not been explored.

Further, it will be most interesting to see how other forms of uncertainty representation, such as general fuzzy measures [25], imprecise probabilities [24], and rough sets [17], can be considered from this perspective.

Finally, the approach presented here is somewhat in the spirit of category theory. It will be interested to compare this approach to that of others exploring the categories of fuzzy systems [1,8].

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