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Challenge problems: uncertainty in system response given uncertain parameters

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Abstract

The risk assessment community has begun to make a clear distinction between aleatory and epistemic uncertainty in theory and in practice. Aleatory uncertainty is also referred to in the literature as variability, irreducible uncertainty, inherent uncertainty, and stochastic uncertainty. Epistemic uncertainty is also termed reducible uncertainty, subjective uncertainty, and state-of-knowledge uncertainty. Methods to efficiently represent, aggregate, and propagate different types of uncertainty through computational models are clearly of vital importance. The most widely known and developed methods are available within the mathematics of probability theory, whether frequentist or subjectivist. Newer mathematical approaches, which extend or otherwise depart from probability theory, are also available, and are sometimes referred to as generalized information theory (GIT). For example, possibility theory, fuzzy set theory, and evidence theory are three components of GIT. To try to develop a better understanding of the relative advantages and disadvantages of traditional and newer methods and encourage a dialog between the risk assessment, reliability engineering, and GIT communities, a workshop was held. To focus discussion and debate at the workshop, a set of prototype problems, generally referred to as challenge problems, was constructed. The challenge problems concentrate on the representation, aggregation, and propagation of epistemic uncertainty and mixtures of epistemic and aleatory uncertainty through two simple model systems. This paper describes the challenge problems and gives numerical values for the different input parameters so that results from different investigators can be directly compared.

Keywords: Epistemic uncertainty; Reducible uncertainty; Subjective uncertainty; Aleatory uncertainty; Variability; Irreducible uncertainty

1. Introduction

Modeling and simulation efforts continue to expand in areas where they have been used for some time, and are also beginning to be used in new application areas. Modeling and simulation have been heavily used in risk assessment for high-consequence systems, such as nuclear power reactors, underground storage of radioactive wastes, and the nuclear weapon stockpile. For complex engineered or natural systems, computer-based modeling and simulation is required. Most of the mathematical models of these physical systems are given by systems of coupled partial differential equations (PDE). For example, typical computer codes that numerically solve these PDEs, and all of their auxiliary equations, can contain tens to hundreds of thousands of lines of code. Similarly, the inputs and outputs of these codes have high dimensionality, for example, hundreds of input and output quantities.

Computer codes of the size and complexity just indicated can be considered as ‘black boxes’ in the sense that very little is known concerning how the codes map inputs to outputs. For the present discussion, we consider this mapping to be deterministic. By deterministic we mean that when all necessary input data for the code are specified, the code produces only one value for every output quantity. For very large codes of interest here, one mapping, i.e. one execution of the code, could require hours or days on even the world’s fastest computers. To be of value to decision makers using the results of the code for risk, reliability, or performance assessment, hundreds or thousands of executions of the code are usually required.

For realistic systems, one must include uncertainties of various types in the mathematical model of the system.

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Uncertainty could occur in parameters in the mathematical model, there could be uncertainty in the accuracy of the mathematical model to describe the system of interest, or in the sequence of possible events that could occur in a discrete event system. During the last 10 or so years, the risk assessment community has begun to make a clear distinction between aleatory and epistemic uncertainty. Aleatory uncertainty is also referred to in the literature as variability, irreducible uncertainty, inherent uncertainty, and stochastic uncertainty. We use the term aleatory uncertainty to describe the inherent variation associated with the physical system or the environment under consideration. Epistemic uncertainty is also termed reducible uncertainty, subjective uncertainty, and model form uncertainty. Epistemic uncertainty derives from some level of ignorance, or incomplete information, of the system or the surrounding environment.

Methods to efficiently represent, aggregate, and propagate different types of uncertainty through computational models are clearly of vital importance. The most widely known and developed methods are available within the mathematics of probability theory, whether frequentist or Bayesian estimation. Newer mathematics, which extend or otherwise depart from probability theory, are also available, and are known collectively by the name generalized information theory (GIT). For example, possibility theory, fuzzy set theory, and evidence theory are three components of GIT (see, for example, Refs. [1–4]).

Assessing and comparing the multiple methods available from probability theory, GIT, and other methodologies has proven to be quite difficult. Moreover, the communities dedicated to risk assessment, reliability engineering, and GIT have not communicated extensively across these disciplines to try to develop a common understanding of the relative advantages and disadvantages of these available methods for problems of different types.

The purpose of this paper is to encourage a dialog between the risk assessment, reliability engineering, and GIT communities on the subject of uncertainty representation, aggregation, and propagation. Our emphasis is on epistemic uncertainty and mixtures of epistemic and aleatory uncertainty. We believe this emphasis is most appropriate because representation, aggregation, and propagation of aleatory uncertainty is well established using traditional probability theory. The chosen mechanism to encourage the dialog is two-fold:

1. Two specific mathematical model systems are described in Section 4. The first is a simple algebraic system, which is, nonetheless, of sufficient complexity to engender significant unanswered questions. The second is a somewhat more complex, but still analytically solvable, dynamical system. It is intended that these two systems serve as simplified models of the kinds of systems that are our real focus.
2. The workshop was sponsored by Sandia National Laboratories in the summer of 2002. National and international leaders of the three communities indicated above, as well as other interested individuals, were invited to participate in the workshop. The two problem sets are intended to serve as a common focus for experimentation, discussion, and comparison of mathematical approaches.

In Section 2, we outline our view of the role of uncertainty representation, aggregation, and propagation within the overall context of computational analysis support for decisions. In Section 3, we briefly discuss the various forms of uncertainty present in such systems, and briefly mention some of the mathematical representations available from the probabilistic, GIT, and other communities. In Section 4, we describe the two problem sets.

2. Simulation, uncertainty, and decision support

The appropriate incorporation of uncertainty into the analyses of complex systems is a topic of importance and widespread interest. The need for such incorporation arises in many contexts. The particular context under consideration here is shown in Fig. 1. We assume to have a mathematical model of a physical system specified. We consider the model to be composed of three major elements: inputs, simulator, and outputs. The inputs are composed of all specifications needed for the simulator to produce one realization of the outputs. A model of the type considered here can be formally represented by the functional formalism: $\mathbf{y} = f(\mathbf{x})$, where $\mathbf{x} = [x_1, x_2, \ldots, x_m]$ is a vector of inputs, f corresponds to the simulator, and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ is a vector of outputs. Examples of inputs in the systems of interest are: parameters representing physical characteristics in the system, the geometric description of the system, and initial and boundary conditions for PDEs. In practice, the dimensionality m and n of $\mathbf{x}$ and $\mathbf{y}$, respectively, can be quite large, for example, one hundred or more. The simulator f is usually given by a large computer code and the cost of one evaluation of f, for a specified $\mathbf{x}$, can be quite high, for example, hours or days on massively parallel computers. One evaluation of f could be considered, for example, as one steady-state solution of a system of PDEs, or it could be considered as the time dependent response of the system. In this latter case, we would have $\mathbf{y}(t) = f(\mathbf{x}; t)$.

The simulator input parameters $\mathbf{x}$ are assumed to be characterized from information given concerning the parameters. The challenge problems, discussed later, directly address issues in the representation and aggregation of information concerning the parameters. The information can be of different types and from a number of sources, including measurements and expert opinion, as discussed in more detail in Section 3. Given a representation and aggregation of the information into the vector $\mathbf{x}$, the vector is propagated through the simulator f. As mentioned above, we assume f to be deterministic; that is, for one realization

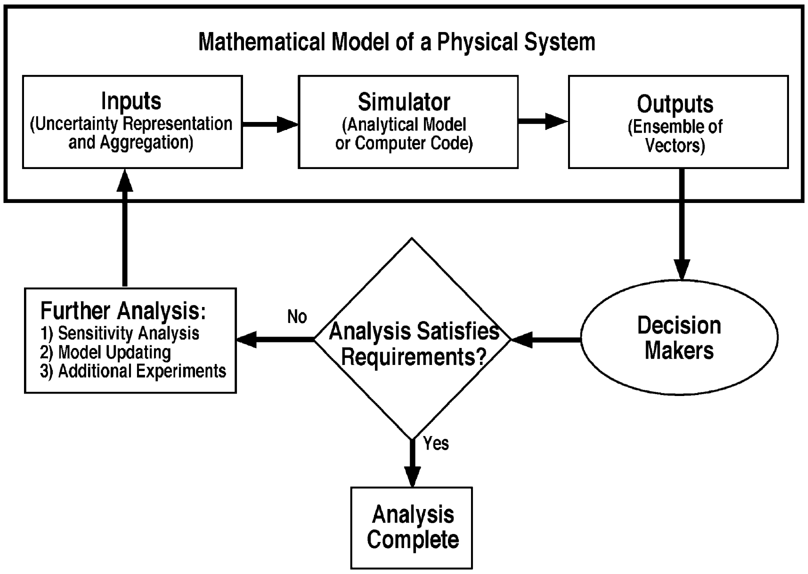

Fig. 1. Propagation of uncertainty through mathematical models in a decision support context.

of **x**, there is only one realization of **y**, but that realization could be a function of space and/or time. Given a representation and aggregation of uncertainty in **x**, and the propagation of the representation through f, we also wish to address the issue of how to interpret the resultant uncertainty representation in **y**.

As indicated in Fig. 1, we assume the analysis of the physical system was conducted for the purpose of making or aiding in a decision or series of decisions. These decisions could be, for example, in engineering design or performance optimization, safety or reliability assessment of an engineered system, or an environmental regulatory decision. Although many analyses may be conducted before the decision maker interprets the results from the analysis, we presume a decision is made with regard to whether the analysis satisfies requirements. We stress that the decision is made on the requirements for the analysis, *not* the requirements for the system performance. For example, the analysis may be adequate to show that the engineered system does not satisfactorily meet performance requirements. Although the analysis is adequate, the decision with regard to the engineered system may be that the system will not be built. A similar example is: given the uncertainties in the system being analyzed, the predicted uncertainty in the system performance may be unacceptably large. As with the previous example, the analysis is adequate, but the system must be redesigned so that predicted uncertainty of the system performance is reduced to an acceptable level. This will be discussed more in Section 3.

If further analysis is required, then different types of activities could be pursued. One of the most common is sensitivity analysis. Whether the physical system satisfies requirements or not, a sensitivity analysis could be conducted to determine which of the uncertainties in **x** results in the largest change in the uncertainties in **y**. The rank order of the effects of the uncertainties in **x** could be with regard to a single element of **y**, or some functional of the elements of **y**. Another type of activity could be the updating of the inputs or the simulator using Bayesian inference. The last example may lead to the decision to conduct additional physical experiments to better determine the uncertainty, or reduce the uncertainty, in selected input quantities. For example, the decision maker may conclude, based on the analysis, that additional information is required for certain physical parameters in **x**. As a result, additional experiments could be conducted to reduce the epistemic uncertainty in these elements of **x**.

The schema presented here, including all of its components, is itself rather complex. While we are aware that the overall context is commonly driven by decision support, we also recognize the difficulty of focusing on all components simultaneously. As a result, we are most interested in concentrating attention on uncertainty representation for inputs, aggregation of uncertainty information from a wide variety of sources, propagation of uncertainty, and uncertainty interpretation in the output. While the participants may choose to deal with the overall decision support context, we strongly encourage participants to address how a method or approach can handle representation, aggregation, propagation, and interpretation of results, in particular.

3. Uncertainty representation and aggregation

Here we briefly discuss some of the available types, sources, and mathematical representations of uncertainty.

3.1. Uncertainty types

One of the most widely recognized distinctions in uncertainty types is between aleatory and epistemic uncertainty. We use the term *aleatory uncertainty* to describe the inherent variation associated with the physical system or the environment under consideration [5–15]. Sources of aleatory uncertainty can commonly be singled out from other contributors to uncertainty by their representation as randomly distributed quantities that can take on values in an established or known range, but for which the exact value will vary by chance from unit to unit or from time to time. The mathematical representation most commonly used for aleatory uncertainty is a probability distribution. When substantial experimental data are available for estimating a distribution, there is no debate that the correct model for aleatory uncertainty is a probability distribution. Propagation of these distributions through a modeling and simulation process is well developed and is described in many texts.

Epistemic uncertainty derives from some level of ignorance of the system or the environment. We use the term *epistemic uncertainty* to describe any lack of knowledge or information in any phase or activity of the modeling process [5–15]. The key feature that this definition stresses is that the fundamental cause is incomplete information or incomplete knowledge of some characteristic of the system or the environment. As a result, an increase in knowledge or information can lead to a reduction in the predicted uncertainty of the response of the system, all things being equal. Examples of sources of epistemic uncertainty are: when there is little or no experimental data for a fixed (but unknown) physical parameter; limited understanding of complex physical processes; and the occurrence of fault sequences or environmental conditions not identified for inclusion in the analysis of the system. As opposed to aleatory uncertainty, the mathematical representation of epistemic uncertainty has proven to be much more of a challenge. In fact, it is our belief that the preeminent issue in uncertainty analysis of systems is the representation, aggregation, and propagation of epistemic uncertainty, as well as mixtures of epistemic and aleatory uncertainty. Our challenge problems have been carefully designed to reflect these fundamental issues.

For any particular physical system of interest that is mathematically modeled, we can distinguish between parametric uncertainty and model form uncertainty. Parametric uncertainty can be entirely aleatoric in nature, for example, stochastic parameters in a specified mathematical model. Model form uncertainty is fundamentally epistemic in nature because it concerns actual structural changes in a model, or selection of one model among a class of models. While parametric and model form uncertainty commonly occur together in realistic analyses, the challenge problems posed here only address parametric uncertainty.

3.2. Types of information sources

It is not unusual for different parametric inputs to be determined from quite different sources and, on that basis, to take very different forms. Some of the types of uncertainty sources that occur in modeling and simulation of physical systems include:

(1) *Strong statistical information*: Sometimes, large quantities of experimental data are available, sufficient to derive or convincingly verify a particular statistical model.
(2) *Sparse statistical information*: More commonly, only a small collection of experimental data points might be available, and collection of further data points might be very expensive or impossible. Further, the available experimental data may provide only indirect or inferential information on the parameters actually used in a particular analysis. In these cases, attempts to fit particular statistical models will leave a substantial residue of epistemic uncertainty.
(3) *Intervals*: Upper and lower bounds or graded levels of belief on parametric values can be provided, typically from expert elicitation. Moreover, when intervals are available, sometimes multiple intervals are provided, for example, from different experts or teams of experts. These collections of intervals together could be coherent, partially contradictory, or even completely contradictory.

Real problems typically present a mixture of such sources across different parameters. Examples of this include: interval data might be available from the sparse output of instrument measurements; the mean of a normal distribution could be elicited from an expert; parameters of another distribution fit with high precision from a large collection of measured point data; and finally, the mean of another distribution may only be presumed to lie within an interval. Also, in real situations, the uncertainty in a given input parameter might be independently estimated from several completely different sources and thus have completely different mathematical representations. Thus, there is the challenge of aggregating these disparate representations into a single representation, which thereby might have a hybrid mathematical form.

Finally, we note that sometimes purely qualitative information is available, for example, provided by experts in the form of linguistic information such as a parameter being 'high' or 'very low,' or a component being 'highly likely to fail.' While we note that there are components of GIT dedicated to representing linguistic uncertainty from such assessments, these are not of interest in this effort.

4. Challenge problem sets

We now introduce our challenge problems and discuss the overall methodology for approaching the problem sets. We wish to foster an open exchange of information among proponents and practitioners of all methods. Consequently, we will neither advocate any particular methodology, nor will we present solutions for the problems stated in Sections 4.3 and 4.4.

4.1. Formation of the problem sets

The problem sets have been crafted to focus on the issues of representation, aggregation, and propagation of uncertainty through mathematical models. We believe that a synthesis is needed of the strategies that have been developed separately within various communities to address the issues embodied in these problems. It is our opinion that if some coherence concerning these simple problem sets can be achieved, then there is some hope that these methods could be extended to realistic target systems. The two problem sets are given by:

1. A simple algebraic system of the form $y = (a + b)^a$
2. A simple dynamic system of the form of an initial value problem given by an linear ordinary differential equation.

Each problem set is intended to serve as a simplified model of the target software applications. In particular, rather than being black boxes, they are both ‘open boxes.’ While the form of Problem Set 1 is simple in the extreme, we believe that, nevertheless, there are substantial questions about how these diverse communities would approach it, given the type of information provided. While Problem Set 2 has a ready analytical solution, we encourage participants to test their methods by considering the numerical solution to the ODE as a black box.

4.2. Challenge format

Before the specific problem sets are given, some comments are appropriate concerning how a dialog among interested colleagues might proceed:

- Each of the problem sets is precisely posed, but the information concerning the parameters is sparse. No further information is available beyond that given and no implications are made for further assumptions. For example, unless specified, we do not state or imply that a parameter is necessarily a random variable, or that any new measurements might be available. We discourage participants from introducing additional assumptions concerning the parameters. However, if this is done, it must be explicitly stated.
- The posed problem sets primarily focus on uncertainty modeling, and not decision support. For example, we do not specify the source or nature of the numerical inputs, nor the nature of the requirements for a decision problem. Participants may choose to address such issues as sensitivity analysis and decision support; however, we request that they focus on uncertainty representation, aggregation, propagation, and results interpretation.
- The main reason for this request is to concentrate the discussion. Another is to reflect the real conditions under which risk analysts and engineering modelers work, and the necessity in some domains to avoid bias by separating the science and mathematics of modeling from the politics or regulatory repercussions of decision making.
- We believe that it is fruitful to approach Problem Set 2 using either the analytical solution to the ODE or a numerical solution to the ODE. In this way, Problem Set 2 would be approached with more of the character of a black box. In other words, one would not assume any knowledge of the system characteristics or response. The only information obtained from the analytical or numerical solution is the mapping of inputs to outputs using sampling-based methods for each time-dependent realization of the system response.

4.3. Algebraic problem set

The form of the mathematical model describing the physical system is known with certitude, i.e. there is no uncertainty about the model form. Only parametric uncertainty in the model is considered. Let y be the system response, and let a and b be continuous parameters in the model. Let the model of the physical process be given by

$$y = (a + b)^a \tag{1}$$

The parameters a and b are independent, i.e. knowledge about the value of one parameter implies nothing about the value of the other. Both a and b are positive real numbers. The task for each problem in the sequence is to quantify the uncertainty in y given the information concerning a and b. Stated differently: what can be ascertained about the response of the system y, given only the stated information concerning a and b?

The sequence of problems begins with very little information concerning a and b so that they are only known to lie within specified intervals. This situation is intended to reflect large epistemic (reducible) uncertainty. Information is added in each subsequent problem in the sequence by way of more specificity concerning the parameters. Information of different types is incrementally added concerning the parameters for each subsequent problem in the sequence. The information given may be mutually supportive, or some of it may be contradictory to some degree. Midway through the sequence of problems, sufficient information is added so that the uncertainty in one of the parameters, b, is characterized by a probability distribution. This type of uncertainty is referred to as aleatory (irreducible) uncertainty.

For each problem in the sequence, only the stated information concerning a and b is available. For an individual example problem, one cannot infer any information from the problem preceding or following it in the sequence. If the stated information concerning a and b

cannot be represented precisely as given, then any assumptions or approximations that are made must be clearly stated.

Six problems are specified in the sequence. The sequence is structured by the type and quantity of information specified for a and b. The structure of the sequence is as follows:

Problem 1: a is in an interval, b is in an interval.
Problem 2: a is in an interval, b is characterized by multiple intervals.
Problem 3: a is characterized by multiple intervals, b is characterized by multiple intervals.
Problem 4: a is in an interval, b is specified by a probability distribution with imprecise parameters.
Problem 5: a is characterized by multiple intervals, b is specified by a probability distribution with imprecise parameters.
Problem 6: a is in an interval, b is a precise probability distribution.

The solution to Problem 1 is straightforward. However, a unique and provably correct solution to Problems 2–6 cannot be demonstrated. The description of each problem in the sequence follows.

Problem 1. a and b are contained in the closed intervals A and B, respectively, where

$$A = [a_1, a_2] \text{ and } B = [b_1, b_2]. \tag{2}$$

Problem 2. a is contained in the closed interval A, and the information concerning b is given by n independent sources of information. Each source specifies a closed interval B_i of possible values for b. One has

$$A = [a_1, a_2] \text{ and } B_i = [b_1^i, b_2^i] \text{ for } i = 1, 2, \ldots, n. \tag{3}$$

All the n sources for b are equally credible. Given this information, consider the following family of problems:

(2a) B_i, $i = 1, 2, \ldots, n$, is a consonant collection of intervals, i.e. the intervals are nested. Without loss in generality, the intervals can be ordered so that

$$B_i \subseteq B_{i+1} \text{ for } i = 1, 2, \ldots, n-1. \tag{4}$$

(2b) B_i, $i = 1, 2, \ldots, n$, is a consistent collection of intervals, i.e. there exists a non-empty interval which is a subinterval of every interval in the collection. This can be written as

$$\cap_i B_i \neq \emptyset. \tag{5}$$

(2c) B_i, $i = 1, 2, \ldots, n$, is an arbitrary collection of intervals, i.e. there is no assumed overlap or relationship among any of the intervals in the collection.

Problem 3. The information concerning a is given by m independent sources of information. Each source specifies a closed interval A_i that contains the value for a. The information concerning b is given by n independent sources of information. Each source specifies a closed interval B_j of possible values for b. One has

$$A_i = [a_1^i, a_2^i] \text{ for } i = 1, 2, \ldots m \text{ and } B_j = [b_1{}^j, b_2{}^j] \text{ for } j = 1, 2, \ldots, n. \tag{6}$$

The m sources for a and n sources for b are all equally credible. Given this information, consider the following family of problems:

(3a) A_i, $i = 1, 2, \ldots, m$, and B_j, $j = 1, 2, \ldots, n$, are consonant collections of intervals, i.e. the intervals for a and b are nested. Without loss in generality, the intervals can be ordered so that

$$A_i \subseteq A_{i+1} \text{ for } i = 1, 2, \ldots, m-1 \text{ and } B_j \subseteq B_{j+1} \text{ for } j = 1, 2, \ldots, n-1. \tag{7}$$

(3b) A_i, $i = 1, 2, \ldots, m$, and B_j, $j = 1, 2, \ldots, n$, are consistent collections of intervals, i.e. for each collection there exists a non-empty interval which is a subinterval of every interval in the collection. This can be written as

$$\cap_i A_i \neq \emptyset \text{ and } \cap_j B_j \neq \emptyset. \tag{8}$$

(3c) $A_i, i = 1, 2, \ldots, m$, and $B_j, j = 1, 2, \ldots, n$, are arbitrary collections of intervals, i.e. there is no assumed overlap or relationship among any of the intervals in the two collections.

Problem 4. a is contained in the closed interval A, and b is given by a log-normal probability distribution. One has

$$A = [a_1, a_2] \text{ and } \ln b \sim N(\mu, \sigma). \tag{9}$$

The value of the mean, μ, and the standard deviation, σ, are given, respectively, by the closed intervals

$$M = [\mu_1, \mu_2] \text{ and } S = [\sigma_1, \sigma_2]. \tag{10}$$

Problem 5. The information concerning a is given by m independent sources of information. Each source specifies a closed interval A_i that contains the value for a. The information concerning b is given by n independent sources of information. Each source agrees that b is given by a log-normal probability distribution, however, each source specifies closed intervals, M_j and S_j, of possible values of the mean μ and the standard deviation σ, respectively. One has

$$A_i = [a_1^i, a_2^i] \text{ for } i = 1, 2, \ldots, m \text{ and } \ln b \sim N(\mu^j, \sigma^j) \text{ for } j = 1, 2, \ldots, n. \tag{11}$$

The value of each mean, μ^j, and standard deviation, σ^j, are given, respectively, by the closed intervals

$$M_j = [\mu_1^j, \mu_2^j] \text{ and } S_j = [\sigma_1{}^j, \sigma_2{}^j] \text{ for } j = 1, 2, \ldots, n. \tag{12}$$

The m sources of information for a are equally credible, as are the n sources for μ and σ. Given this information, consider the following family of problems:

(5a) A_i, $i = 1, 2, \ldots, m$, is a consonant collection of intervals, i.e. the intervals are nested. Without loss in generality, the intervals can be ordered so that

$$A_i \subseteq A_{i+1} \text{ for } i = 1, 2, \ldots, m-1. \tag{13}$$

Each of M_j and S_j, $j = 1, 2, \ldots, n$, is consonant collections of intervals, i.e. the intervals are nested. Without loss in generality the intervals can be ordered so that

$$M_j \subseteq M_{j+1} \text{ and } S_j \subseteq S_{j+1} \text{ for } j = 1, 2, \ldots, n-1. \tag{14}$$

(5b) A_i, $i = 1, 2, \ldots, m$, is a consistent collection of intervals, i.e. there exists a non-empty interval which is a subinterval of every interval in the collection. This can be written as

$$\cap_i A_i \neq \emptyset. \tag{15}$$

Each of M_j and S_j, $j = 1, 2, \ldots, n$, is consistent collections of intervals, i.e. for each collection there exists a non-empty interval which is a subinterval of every interval in the collection. This can be written as

$$\cap_j M_j \neq \emptyset \text{ and } \cap_j S_j \neq \emptyset. \tag{16}$$

(5c) A_i, $i = 1, 2, \ldots, m$, is an arbitrary collection of intervals, i.e. there is no assumed overlap or relationship among any of the intervals in the collection. Each of M_j and S_j, $j = 1, 2, \ldots, n$, is an arbitrary collections of intervals, i.e. there is no assumed overlap or relationship among any of the intervals in the two collections.

Problem 6. a is contained in the closed interval, A, and b is given by a log-normal probability distribution. One has

$$A = [a_1, a_2] \text{ and } \ln b \sim N(\mu, \sigma). \tag{17}$$

The values of the mean μ and the standard deviation σ are precisely known.

Appendix A contains the suggested numerical values for each problem so that solutions from different participants can be directly compared.

4.4. ODE problem

Consider a simple linear oscillator given by the mass-spring-damper system acted on by the forcing function $Y \cos \omega t$, shown in Fig. 2. The displacement and the velocity of the mass relative to a fixed reference frame are given by x and $\dot{x}$, respectively. The equation of motion for the mass is given by:

$$m\ddot{x} + c\dot{x} + kx = Y \cos \omega t \tag{18}$$

where m is the mass of the cart, k is the spring constant of the linear spring, and c is the damping coefficient of the linear damper. Y and ω are the amplitude and frequency, respectively, of the excitation acting on the mass. The initial conditions for Eq. (22) are precisely known to be $x = \dot{x} = 0$. After imposing the specified initial conditions,

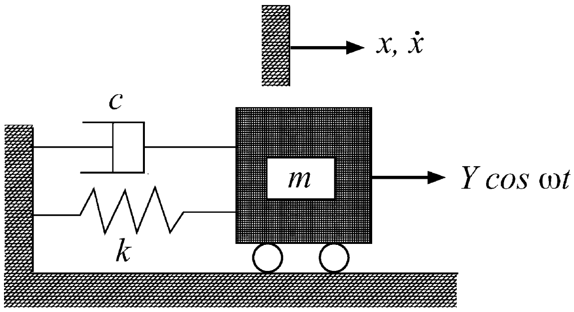

Fig. 2. Mass-spring-damper system acted on by an excitation function.

the analytical solution to Eq. (22) is given by [16]

$$x(t) = (A\cos\omega_d t + B\sin\omega_d t)\exp\left(\frac{-ct}{2m}\right) + \frac{Y}{C}\cos(\omega t - \alpha) \tag{19}$$

where

$$A = -\frac{Y}{C}\cos\alpha \tag{20}$$

$$B = -\frac{Y}{C\omega_d}\left(\omega\sin\alpha + \frac{c}{2m}\cos\alpha\right) \tag{21}$$

$$C = \sqrt{(k - m\omega^2)^2 + (c\omega)^2} \tag{22}$$

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{23}$$

$$\alpha = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{24}$$

ω_d is the damped natural frequency of the system and α is the phase angle of the steady-state response relative to the excitation function. Note that for α the principle value of the arctangent function is the value of interest.

The parameters m, k, c, Y, and ω are independent, i.e. knowledge about the value of one parameter implies nothing about the value of the other. The five parameters are all positive real numbers. The task for this problem is to quantify the uncertainty in the steady-state magnification factor D_s, given only the stated information for the problem. D_s is defined as the ratio of the amplitude of the steady-state response of the system to the static displacement of the system induced by a force of magnitude Y. Therefore, one has

$$D_s = \frac{X}{Y/k} \tag{25}$$

where X is the amplitude of the steady-state response of the system, and Y/k is the static displacement of the system induced by a force of magnitude Y. The steady-state magnification factor can be analytically derived as [16]

$$D_s = \frac{k}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \tag{26}$$

If the stated information concerning the parameters m, k, c, Y, and ω cannot be represented precisely as given, then any assumptions or approximations that are made must be clearly stated. For example, if a parameter is given as a single closed interval, and the assumption is made that the value is a random variable occurring within the interval, then this assumption must be clearly stated.

The information for each parameter is given in the following discussion.

Parameter m: m is given by a triangular probability distribution defined on the interval $[m_{\min}, m_{\max}]$ with a mode of m_{mod}. The values of $m_{\min}$, m_{mod}, and $m_{\max}$ are precisely known.

Parameter k: The information concerning k is given by n independent sources of information. Each source agrees that k is given by a triangular probability distribution; however, each source specifies closed intervals, Min_i, Mod_i, and Max_i of possible values of the minimum, $k_{\min}$, mode, k_{mod}, and maximum value, $k_{\max}$, respectively. The relationship between these values is

$$k_{\min} \le k_{\text{mod}} \le k_{\max} \text{ and } k_{\min} < k_{\max} \quad (27)$$

The values of $k_{\min}, k_{\text{mod}}$, and $k_{\max}$ are contained, respectively, in the closed intervals

$$\text{Min}_i = [a_1^i, a_2^i], \quad \text{Mod}_i = [h_1^i, h_2^i] \text{ and } \text{Max}_i = [b_1^i, b_2^i] \text{ for } i = 1, 2, \ldots, n. \quad (28)$$

The n sources of information for k are equally credible. $\text{Min}_i, \text{Mod}_i$, and Max_i are consistent collections of intervals, i.e. each collection has a non-empty interval which is a subinterval of every interval in the collection. This can be written as

$$\cap_i \text{Min}_i \ne \emptyset, \quad \cap_i \text{Mod}_i \ne \emptyset, \text{ and } \cap_i \text{Max}_i \ne \emptyset. \quad (29)$$

Parameter c: The information concerning c is given by q independent sources of information. Each source specifies a closed interval C_j of possible values for c. One has

$$C_j = [c_1^j, c_2^j] \text{ for } j = 1, 2, \ldots q. \quad (30)$$

All of the q sources for C are equally credible. C_i is an arbitrary collection of intervals, i.e. there is no assumed overlap or relationship among any of the intervals in the collection.

Parameter Y: Y is contained in the closed interval $\mathbf{Y}$, where

$$\mathbf{Y} = [Y_1, Y_2]. \quad (31)$$

Parameter ω: ω is given by a triangular probability distribution defined on the interval $[\omega_{\min}, \omega_{\max}]$ with a mode of ω_{mod}. The values of $\omega_{\min}, \omega_{\text{mod}}$, and $\omega_{\max}$ are contained, respectively, in the closed intervals

$$\text{Min} = [\alpha_1, \alpha_2], \ \text{Mod} = [\eta_1, \eta_2] \text{ and } \text{Max} = [\beta_1, \beta_2]. \quad (32)$$

The relationship between these values is

$$\omega_{\min} \le \omega_{\text{mod}} \le \omega_{\max} \text{ and } \omega_{\min} < \omega_{\max} \quad (33)$$

Appendix B contains the suggested numerical values for each parameter so that solutions from different participants can be directly compared.

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Appendix A. Numerical values for the algebraic problem set

Problem 1

$A = [0.1, 1.0]$,

$B = [0.0, 1.0]$.

Problem 2a

$A = [0.1, 1.0]$,

$B_1 = [0.6, 0.8]$, $B_2 = [0.4, 0.85]$, $B_3 = [0.2, 0.9]$, and $B_4 = [0.0, 1.0]$.

Problem 2b

$A = [0.1, 1.0]$,

$B_1 = [0.6, 0.9]$, $B_2 = [0.4, 0.8]$, $B_3 = [0.1, 0.7]$, and $B_4 = [0.0, 1.0]$.

Problem 2c

$A = [0.1, 1.0]$,

$B_1 = [0.6, 0.8]$, $B_2 = [0.5, 0.7]$, $B_3 = [0.1, 0.4]$, and $B_4 = [0.0, 1.0]$.

Problem 3a

$A_1 = [0.5, 0.7]$, $A_2 = [0.3, 0.8]$, $A_3 = [0.1, 1.0]$,

$B_1 = [0.6, 0.6]$, $B_2 = [0.4, 0.85]$, $B_3 = [0.2, 0.9]$, and $B_4 = [0.0, 1.0]$.

Problem 3b

$A_1 = [0.5, 1.0]$, $A_2 = [0.2, 0.7]$, $A_3 = [0.1, 0.6]$,

$B_1 = [0.6, 0.6]$, $B_2 = [0.4, 0.8]$, $B_3 = [0.1, 0.7]$, and $B_4 = [0.0, 1.0]$.

Problem 3c

$A_1 = [0.8, 1.0]$, $A_2 = [0.5, 0.7]$, $A_3 = [0.1, 0.4]$,

$B_1 = [0.8, 1.0]$, $B_2 = [0.5, 0.7]$, $B_3 = [0.1, 0.4]$, and $B_4 = [0.0, 0.2]$.

Problem 4

$A = [0.1, 1.0]$,

$M = [0.0, 1.0]$, and $S = [0.1, 0.5]$.

Problem 5a

$A_1 = [0.5, 0.7]$, $A_2 = [0.3, 0.8]$, $A_3 = [0.1, 1.0]$,

$M_1 = [0.6, 0.8]$, $M_2 = [0.2, 0.9]$, $M_3 = [0.0, 1.0]$,

$S_1 = [0.3, 0.4]$, $S_2 = [0.2, 0.45]$, and $S_3 = [0.1, 0.5]$.

Problem 5b

$A_1 = [0.5, 1.0]$, $A_2 = [0.2, 0.7]$, $A_3 = [0.1, 0.6]$,

$M_1 = [0.6, 0.9]$, $M_2 = [0.1, 0.7]$, $M_3 = [0.0, 1.0]$,

$S_1 = [0.3, 0.45]$, $S_2 = [0.15, 0.35]$, and $S_3 = [0.1, 0.5]$.

Problem 5c

$A_1 = [0.8, 1.0]$, $A_2 = [0.5, 0.7]$, $A_3 = [0.1, 0.4]$,

$M_1 = [0.6, 0.8]$, $M_2 = [0.1, 0.4]$, $M_3 = [0.0, 1.0]$,

$S_1 = [0.4, 0.5]$, $S_2 = [0.25, 0.35]$, and $S_3 = [0.1, 0.2]$.

Problem 6

$A = [0.1, 1.0]$,

$\mu = 0.5$, $\sigma = 0.5$.

Appendix B. Numerical values for the ODE problem

Values for m

$m_{\min} = 10$, $m_{\text{mod}} = 11$, and $m_{\max} = 12$

Values for k

$\text{Min}_1 = [90, 100]$, $\text{Min}_2 = [80, 110]$, $\text{Min}_3 = [60, 120]$

$\text{Mod}_1 = [150, 160]$, $\text{Mod}_2 = [140, 170]$, $\text{Mod}_3 = [120, 180]$

$\text{Max}_1 = [200, 210]$, $\text{Max}_2 = [200, 220]$, $\text{Max}_3 = [190, 230]$

Values for c

$C_1 = [5, 10]$, $C_2 = [15, 20]$, $C_3 = [25, 25]$

Values for Y

$Y = [70, 100]$

Values for ω

$\text{Min} = [2, 2.3]$, $\text{Mod} = [2.5, 2.7]$, $\text{Max} = [3.0, 3.5]$

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