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Understanding high-order network structure using permissible walks on attributed hypergraphs

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ABSTRACT

Hypergraphs have been a recent focus of study in mathematical data science as a tool to understand complex networks with high-order connections. One question of particular relevance—how to leverage information carried in hypergraph attributions when doing walk-based techniques. In this work, we focus on a new generalization of a walk in a network that recovers previous approaches and allows for a description of permissible walks in hypergraphs. Permissible walk graphs are constructed by intersecting the attributed s-line graph of a hypergraph with a relation respecting graph. The attribution of the hypergraph's line graph commonly carries over information from categorical and temporal attributions of the original hypergraph. To demonstrate this approach on a temporally attributed example, we apply our framework to a Reddit data set composed of hyperedges as threads and authors as nodes where post times are tracked.

KEYWORDS: hypergraphs; network science; higher-order networks; temporal graphs; graph walks; attributed graphs.
Math Subject Classification: 05C65; 05C76; 05C20.

1. INTRODUCTION

Graph theory has a rich history in the analysis of complex systems as network models with a set of entities (nodes) with pairwise relationships (edges). Graph models benefit from their simplicity and swath of supporting theory. However, many real-world systems are not accurately modeled with just a dyadic connection between nodes, as in a graph data structure. Graphs naturally generalize to structures known as hypergraphs, in which a set of k nodes that are interacting can instead be modeled by a size k hyperedge, for $k \geqslant 1$. However, it is notably difficult to generalize many graph-theoretic notions and techniques, for example spectral analysis [1–3] and random walks [4], to the

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world of hypergraphs. Instead, a common approach to study hypergraphs is by using other graphs which approximate them to capture the higher-order connections between hyperedges.

Additionally, it is common that both graphs and hypergraphs carry data, called attributes, in the form of numerical or vector weights, or other data types such as Boolean, categorical, or ordinal data. Such attributes can be assigned to vertices, edges, hyperedges, or some combination of these, depending on the application. They can arise from measured data, or can be calculated from other data sources or from inherent properties of the graph itself.

In this work, we present a general methodology for approaching walks in hypergraphs that respects a relation on its attributes. In this method, we create a new graph representation of the hypergraph that summarizes the structure of the underlying hypergraph while respecting the attributes. This approach is able to capture both the connectivity of the hypergraph as well as how the attributed information is related.

An area of likely use for such a graph structure would be in the study of its traversals, which is a classical area of interest in graph theory. One type of traversal is a *walk*. Given some vertex u in a graph, we can "walk" from u to another vertex v if u and v are connected by an edge. A walk is then a sequence of vertices $v_1 v_2 \ldots v_k$ such that each v_i and v_{i+1} are connected by an edge. These walks are used in many applications with a major one being in vectorization of nodes (e.g. node2vec [5]), in which data scientists can study the relations between nodes with applications such as classification through clustering and training natural language processing models.

Graph walks are readily understood in application, but they rely on the structure of graphs in such a way that needs to be extended to the language of hypergraphs carefully [4, 6, 7]. In a graph, if two vertices are connected by an edge, they are connected by a unique edge (considering only a simple graph and not a multi-graph). So, when a graph walk moves from node u to node v, we implicitly understand the edge through which we traverse. Indeed, describing a graph walk in terms of a sequence of vertices, a sequence of edges, or an alternating sequence and vertices and edges, are all equivalent. But in a hypergraph, two vertices can both be contained in any number of edges. Recently in the literature there has been a keen interest in defining hypergraph walks in ways that can be recovered by a graph structure while still incorporating information from the hypergraph. One such approach are s-walks [8], where an s-line graph (see Definition 2.2) is constructed from the hypergraph.

There has also been an active community focused on walks in graphs that incorporate temporal information [9, 10] as many graphs of interest may evolve over time with edges and nodes having temporal attributes. These call for the development of a mathematical framework of dynamic network science known as temporal networks [11]. Such structures arise in a variety of applied settings, be it social network analysis [12], transit network analysis [13], biological interaction networks such as disease spread [14], or communication networks for space satellites [15].

In this work, we study walks in attributed hypergraphs by defining the permissible walk graph of a hypergraph and use it to generalize s-walks to attributed hypergraphs. In doing so we transfer rich attribute information from the hypergraph to the s-line graph structure that would normally be lost in the s-walks framework. This allows for more accurate modeling of data and the resulting application of the corresponding random walks (e.g. node and hyperedge embedding). Some common attributes that can be found on hypergraphs are temporal and type attributes, where, for example, a hyperedge may only be active during a specific interval of time or the edge may be of a specific type that can only communicate to another hyperedge of that type. The goal of this work is to incorporate this information into the s-line graph and show how this better models complex hypergraph data. We demonstrate this through a worked example with temporal attributes.

1.1 Related work

In [16] a similar procedure to our initial exhibition in [17] was developed in which they take a higher-order network (hypergraph) and construct an attributed stream graph. Our work here again is a generalization of this method based on their modeling of only temporal attributes of higher

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order interactions. In [18–20] a method developed on the line-graph of a dynamic multi-graph was used for edge clustering. This is similar to our method as it takes a dynamic multi-graph, which is an attributed hypergraph with direction and temporal attributes, and creates an attributed line-graph. However, our method is a generalization of this approach as it can take higher-order networks as well as multiple attribute types.

1.2 Example in Reddit

To give a glimpse into the importance of considering attributes, in Fig. 1, we show both the s-line graph and our permissible walk graph of a temporally attributed hypergraph modeling Reddit communication between two subreddits: China_Flu and COVID19. In Fig. 1a, we show the s-line graph of the underlying hypergraph for $s = 10$ where the nodes in the graph represent the hyperedges of the underlying hypergraph where hyperedges are threads that are composed of sets of authors (only threads with at least 400 authors are used for visualization purposes). Each edge in this graph is bi-directional as the temporal attributes are not considered in this model. In comparison, in Fig. 1c, we demonstrate the permissible walk graph of the same attributed hypergraph but now the edges follow the temporal order. Specifically, we add directed edges between nodes (threads) if a thread was active before another thread. By doing this we are able to capture the flow of thread creation and subreddit creation that would normally be lost when just considering the s-line graph connectivity. We quantify this as shown in Fig. 1b and d, where we count the number of edges between the subreddits and represent it as a matrix. Figure 1b is for the s-line graph where the interaction is symmetric since each edge is bidirectional. This graph loses the temporal flow of thread and subreddit creation. In comparison, Fig. 1d shows this interaction matrix for the permissible walk graph which is asymmetric with more edges going from China_Flu to COVID19 capturing the temporal directionality of the subreddit creation and activity.

This example demonstrates the importance of incorporating attribute information in the permissible walk graph of an attributed hypergraph when tracking the evolution of a temporal hypergraph. Additionally, it provides more information on the flow of conversations between these subreddits that is normally lost when using the s-line graph.

1.3 Organization

This work begins by introducing the permissible walk graph in Section 2 and explaining how it is formed from an attributed hypergraph. In Section 3 we overview many of the details for implementing the permissible walk graph for studying attributed hypergraphs, including two examples illustrating possible relations defined on the attributes. In Section 4 we use the permissible walk graph to study a Reddit social network alluded to in Fig. 1 where we in detail describe some of the benefits of studying such a dataset from the permissible walk framework. Lastly, in Section Conclusion we provide concluding remarks and future research for permissible walk graphs.

2. METHOD

In this section we present our methodology for constructing a permissible walk graph from an attributed hypergraph. The general procedure is illustrated in Fig. 2 going from an attributed hypergraph to an attributed s-line graph, and lastly the permissible walk graph. Following this section we introduce a small example in Section 3 to better demonstrate the implementation of the permissible walk graph for studying an attributed hypergraph.

Following Fig. 2, we begin by defining an attributed hypergraph.

Definition 2.1 Let $H = (V, E, \phi, \varepsilon, \gamma)$ be an **attributed hypergraph**, where:

- $V = \{v_i\}_{i=1}^{n}$ is a finite, non-empty set of **vertices**;
- $E = \{e_j\}_{j=1}^{m}$ is a finite, non-empty set of **edges** or **hyperedges** $e_j \subseteq V$;
- $\phi : V \to X$ is the **vertex attribute function** to a set of vertex attributes X;

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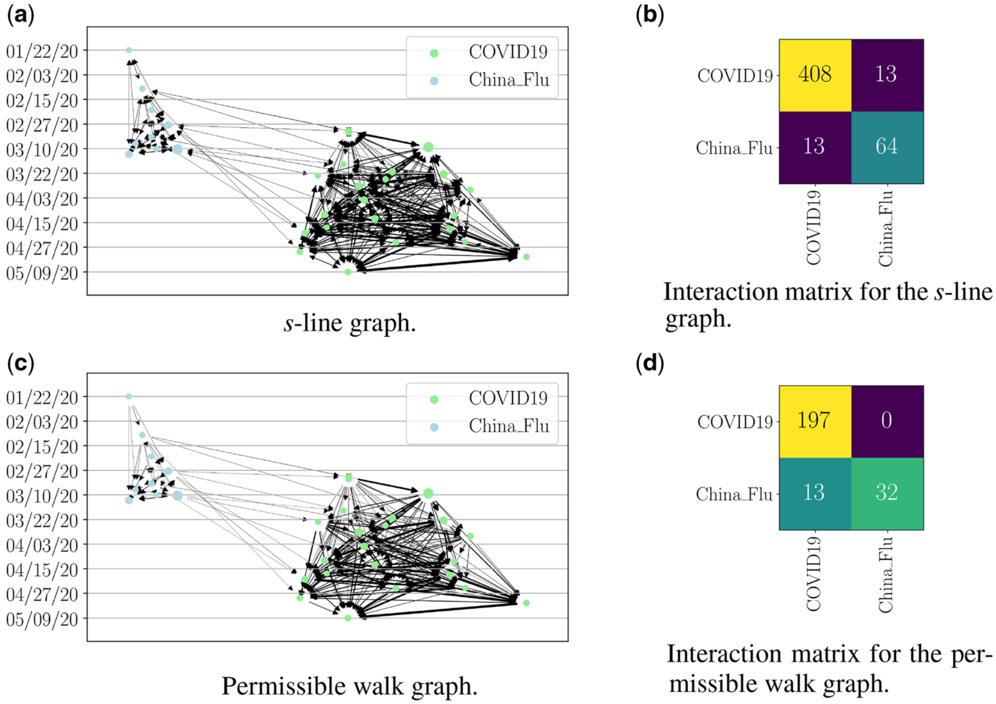

(a) *s*-line graph.

(b) Interaction matrix for the *s*-line graph.

(c) Permissible walk graph.

(d) Interaction matrix for the permissible walk graph.

Figure 1. Comparison between the interaction matrices in (b) and (d) of the *s*-line graph in (a) and permissible walk graph in (c), respectively, from the attributed hypergraph associated to the China_Flu and COVID19 subreddits of Reddit.
Alt text: The figure compares two network representations of Reddit communication between the China_Flu and COVID19 subreddits: the s-line graph and the permissible walk graph. The top row displays the s-line graph (a), showing undirected connections between threads sharing authors, and its symmetric interaction matrix (b) quantifying these connections between subreddits. The bottom row illustrates the permissible walk graph (c), with directed edges representing the temporal order of thread activity, and its asymmetric interaction matrix (d) showing the directed flow of communication. The permissible walk graph and its interaction matrix reveal a temporal directionality, with a stronger flow from China_Flu to COVID19, which is not captured by the symmetric s-line graph, highlighting the importance of considering temporal attributes in network analysis.

- $\varepsilon : E \to Y$ is the **edge attribute function** to a set of edge attributes Y; and
- $\gamma : I \to Z$ is the **incidence attribute function**, where $I \subseteq V \times E$ is the collection of all incidences (i.e. $(v, e) \in R$ if $v \in e$, see discussion below), and Z is a set of incidence attributes.

To explain the use of the term incidence attribute function above, every hypergraph H yields a binary relation $I \subseteq V \times E$ between the vertex set V and the edge set E where if $v \in e$ then vIe, which is the domain of γ . One could also define a Boolean **incidence function** $I' : V \times E \to \{0, 1\}$, where $I'(v, e) = 1$ if and only if $v \in e$, or a Boolean **incidence matrix** I'', where rows represents elements of V, columns represent elements of E and $I''[v_i][e_j] = 1$ if and only if $v_i \in e_j$. In a slight abuse of notation, we will use the symbol I for all three of the incidence relation, function and matrix. The pair $(v_i, e_j) \in I$ is called an individual **incidence**. We call the vertices $v_i \in V$ and the edges $e_j \in E$ generically **objects** (Note that our definitions also operate for the dual hypergraph. Viewing I as the incidence matrix we get a natural way to define the **dual** hypergraph H^* which has E as its vertex set V as its edge set and I^T as its incidence matrix, if H is attributed then the attributes carry to the dual in a natural way with $H^* = (E, V, \varepsilon, \phi, \gamma^*)$ where we let $I^* \subseteq E \times V$ be the subset such that

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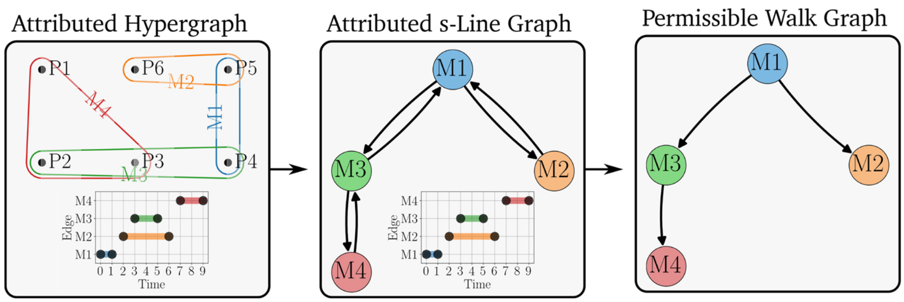

Figure 2. The pipeline for generating the permissible walk graph starting with an attributed hypergraph, which is used to construct the attributed *s*-line graph using an attribute transferring function. The permissible walk graph is then constructed as an attribute respecting subgraph of the *s*-line graph.
Alt text: The figure illustrates the pipeline for constructing a permissible walk graph from an attributed hypergraph. The process begins with an attributed hypergraph, which serves as the input for creating an attributed s-line graph. This transformation utilizes an attribute transferring function to map attributes from the hypergraph to the edges (which become nodes) of the s-line graph. Finally, the permissible walk graph is derived as an attribute-respecting subgraph of this attributed s-line graph, indicating that only edges satisfying certain attribute-based conditions are included in the final graph.

$(e, v) \in I^*$ if $e \ni v$, and $\gamma^* : I^* \to Z$, where $\gamma^*(e_i, v_j) = \gamma(v_i, e_j)$. When considering duals we call the original H the **primal** and H^* the dual, noting that this is arbitrary. For the remainder of this work we operate in the primal pointing of the attributed hypergraphs.).

Following the procedure in Fig. 2 we next construct the attributed *s*-line graph by first building the underlying *s*-line graph.

Definition 2.2 For $s \in \mathbb{Z}_{\geq 0}$, the ***s*-line graph** $L_s = (E, \hat{K})$ of a hypergraph $H = (V, E)$ is a simple graph constructed as follows:

1. The node set E of L_s is the set of hyperedges E of H.
2. $\hat{K} \subseteq \binom{E}{2}$ is a set of undirected graph edges between the vertices $e_i \in E$, where for each pair of hyperedges $e_i \neq e_{i'} \in E$, there exists an edge $\{e_i, e_{i'}\} \in \hat{K}$ if and only if $|e_i \cap e_{i'}| \geq s$.

For example, given the hypergraph in Fig. 2 we can construct its *s*-line graph at various levels of *s* to capture the strength of the hyperedges connections as shown in Fig. 3.

The attributed line graph $\mathcal{L}_s$ is derived from the line graph L_s and the attributes of the attributed hypergraph $H = (V, E, \phi, \varepsilon, \gamma)$, but has bidirected edges K:

Definition 2.3 The **attributed *s*-line graph** $\mathcal{L}_s = (E, K, \tau, \zeta)$ of an attributed hypergraph $H = (V, E, \varepsilon, \phi, \gamma)$ has node set E, but now $K \subseteq E^2$ is a set of directed edges, with both $(e_i, e_{i'}), (e_{i'}, e_i) \in K$ if $|e_i \cap e_{i'}| \geqslant s$, for $e_i \neq e_{i'}$. Here $\to A$ is a vertex attribute function to the attribute set A and $\zeta : K \to B$ is an edge attribute function to the edge attribute set B.

This definition is quite general. But in this work we simply use $A = Y$ and $\tau = \varepsilon$, so that the vertex attributes of the line graph $\mathcal{L}_s$ are simply the edge attributes of the source hypergraph H. Note also that the edge attribute function ζ of $\mathcal{L}_s$ need not have a direct representation in the hypergraph, as it is on *ordered pairs* of *s*-incident edges. While available in our formalism, in this paper we will not use ζ further.

Our last step is to generate the permissible walk graph from the attributed *s*-line graph by considering relations on the attributes.

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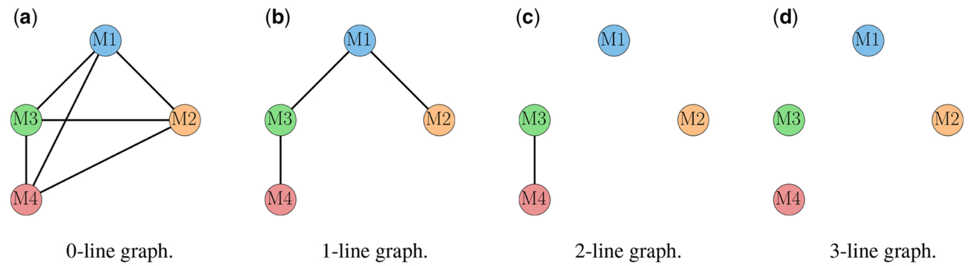

Figure 3. *s*-line graphs for the hypergraph in Fig. 2 for with undirected edges with s values of 0, 1, 2, 3 in (a), (b),(c), and (d), respectively.
Alt text: The figure presents four subplots illustrating s-line graphs of a hypergraph for s∈ {0,1,2,3}. Each subplot displays a graph where nodes represent hyperedges, and undirected edges connect them if they share at least s vertices. The 0-line graph (s = 0) shows a complete graph, the 1-line graph (s = 1) displays fewer connections based on sharing at least one vertex, the 2-line graph (s=2) further reduces connections requiring at least two shared vertices, and the 3-line graph (s=3) exhibits the sparsest graph with connections only for hyperedges sharing three or more vertices. Arranged horizontally, the subplots demonstrate the progressive decrease in connectivity of s-line graphs as the value of s increases, derived from the hypergraph in Fig. 1.

Definition 2.4 For an attributed hypergraph $H = (V, E, \varepsilon, \phi, \gamma)$, let $q : A \times A \to \{0, 1\}$ be called a **predicate**. Then given an attributed s-line graph $\mathcal{L}_s = (E, K, \tau, \zeta)$, the **permissible walk graph** $P_q = (E, Q, \tau, \zeta|_Q)$ for a predicate q is a spanning subgraph of $\mathcal{L}_s$ with edge set $Q \subseteq K$, where Q contains all edges $(e_j, e_j') \in K$ satisfying $q(\tau(e_j), \tau(e_{j'})) = 1$. That is, $Q = \{(e_j, e_{j'}) \in K \colon q(\tau(e_j), \tau(e_{j'})) = 1\}$.

Given a predicate $q : A \times A \to \{0, 1\}$, we can define a directed graph R_q with vertex set E (the same vertex set as the line graph L_s), and (e_j, e_j') is an edge of R_q if $q(\varepsilon(e_j), \varepsilon(e_j')) = 1$, i.e. if the edge attributes of e_j and e_j' are mapped to 1 under q. We will call this graph R_q the **attribution graph**. One way to understand the permissible walk graph is that the perissible walk graph is the (graph) intersection of R_q and $\mathcal{L}_s$, with vertex and edge atributes inherited from $\mathcal{L}_s$. However, in practice, it is more computationally efficient to compute the permissible walk graph by computing Q from just the edge set K. Additionally, the operations of predicate and graph intersections commute.

Notice as well that where the edge set K in the attributed line graph $\mathcal{L}_s$ is bidirected in that each pair of vertices is connected by a dual pair of directed edges, one in each direction, after intersection with the attribution graph R_q, the pairs of vertices in the permissable walk graph P_q may be connected by directed edges in $Q \subseteq K$ in either direction, both directions, or not at all.

While relations and predicate functions can be defined on edge, incidence, or vertex attributes, it may be more useful to associate (or transfer) the attributes from one attribute location to another. For example, we might have vertex or edge attribute functions be induced from a given incidence attribute function, which we refer to as vertex (resp. edge) **marginalization**. For another example, we might use given vertex and edge attribute function to create an induced incidence function, which we refer to as an **extension**. We demonstrate a marginalization function in our toy example in Section 3.1.

Above we discussed how we would generally be ignoring the possibility of edge attributes B on the vertices of the attributed line graph $\mathcal{L}_s$ (which vertices represent *pairs* of incident edges in the source attributed hypergraph H). But if there were a relevant edge attribution function ζ the predicate function could also be defined on the edge attributes where $q : B \to \{0, 1\}$. An example of how this construction could be useful is if $\zeta(e_j, e_{j'}) = |e_j \cap e_{j'}|$. In this case B is the natural numbers and if for $k \in B$ we have $q(k) = 1$ if $k \geq s$ and 0 otherwise we can get the s-line graph as a permissible walk graph on the line graph of the hypergraph. This shows that the permissible walk framework is strictly

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more general than the s-line framework as the s-line graph can be generated from our permissible walk graph. However, we build all of our permissible walk graphs on top of the s-line framework for technical simplicity since it is incorporated into all of our later examples.

The resulting permissible walk graph P_q captures information on both the strength of connectivity between hyperedges through the s parameter as well as specific relations desired between hyperedges.

3. INSTANTIATIONS AND CASES

In this section, we provide two examples demonstrating the application of the permissible walk graph as well as several implementation details.

While an attributed hypergraph can have any type of hyperedge attributes, our method requires that a predicate q is defined on the set of vertex attributes A. Some common attribute types that would allow for a relation to be defined are as follows:

- **Logical attributes**: The attribute functions can take values in the set $\{0, 1\}$. Relations and the corresponding predicate function can then be defined based on logical expressions (e.g. and/or).
- **Semantic type hierarchy attributes**: The attributes are elements of a hierarchical data structure, for example a semantic class hierarchy as used in a computational ontology [21], which has an underlying poset. The relation can then be implied by the hierarchy.
- **Set attributes**: For the set attribute type, consider ϕ, ε, and γ taking values in a set without any additional structure applied. We can define a relation on these sets using an intersection predicate such that they are related if they intersect by at least t elements of the sets. We consider this attribute type in our toy example with sets attributed on each incidence as shown in Fig. 4c.
- **Temporal attributes**: For temporal attributes, the functions ϕ, ε, and γ have temporal intervals (start time less than or equal to end time) as the attribute. Relations between intervals can be defined in many ways including weak, strong, or inclusion orders [22]. An example of interval attributes on the edges as $\varepsilon : E \to Y$ is shown in Fig. 4b of our toy example.

A common occurrence in attributed hypergraphs is multiple attribute types existing on the edges (e.g. time and logical attributes). In this case given a relation of interest on each attribute type one can establish a natural relation on all pairs of these attribute types by asking that both relations are satisfied by their respective attributes. This construction yields the same permissible walk graph as constructing one for each attribute and taking their intersection; however, it is notationally burdensome. In light of this in the following examples when there is more than one attribute present the authors opt to create a permissible walk graph for each attribute and then intersect them to obtain a permissible walk graph over both attributes.

3.1 Toy example

We now build our first example hypergraph, which we use to motivate and demonstrate various aspects and uses of the permissible walk graph. Code for replicating this toy example will be available through the permissible walks module of HyperNetX. This toy example attributed hypergraph in Fig. 4a is constructed to model four meetings, $M_1 \ldots, M_4$, that occurred between six people, $P_1, \ldots, P_6$, with a total of six topics discussed, $A, B, \ldots, F$. Let $\mathcal{A} := \{A, B, \ldots, F\}$. These meetings consisted of subsets of people which have both topic and temporal attributes. Specifically, we placed interval attributes on the edges as shown in Fig. 4b and topic attributes on the incidences as shown in Fig. 4c. The incidence attributes Z of topics in Fig. 4c represent which topics were discussed by which people in each meeting (e.g. person P_4 discussed topics A and B in meeting M_1) and the interval attributes in Fig. 4b describe when the meeting occurred (e.g. M_1 occurred from time 0

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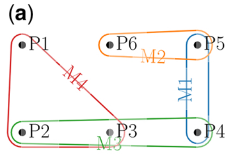
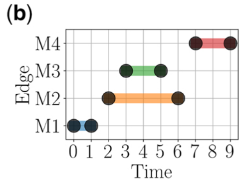


Hypergraph of meetings.



Time interval edge attributes.

(c)

		$\{A,B,C\}$ M_1	$\{E,F\}$ M_2	$\{B,C,D\}$ M_3	$\{C\}$ M_4
$\{C\}$	P_1	$\emptyset$	$\emptyset$	$\emptyset$	$\{C\}$
$\{C,D\}$	P_2	$\emptyset$	$\emptyset$	$\{C,D\}$	$\{C\}$
$\{B,C\}$	P_3	$\emptyset$	$\emptyset$	$\{B,C\}$	$\{C\}$
$\{A,B,D\}$	P_4	$\{A,B\}$	$\emptyset$	$\{D\}$	$\emptyset$
$\{A,C,E,F\}$	P_5	$\{A,C\}$	$\{E,F\}$	$\emptyset$	$\emptyset$
$\{F\}$	P_6	$\emptyset$	$\{F\}$	$\emptyset$	$\emptyset$

Topic incidence attributes.

Figure 4. Attributed hypergraph of meetings in (a) with meeting time intervals as edge attributes in (b) and topics as incidence attributes in (c).

Alt text: The figure illustrates an attributed hypergraph representing four meetings (M1-M4) of six people (P1-P6) discussing six topics (A-F). Subfigure (a) provides a visual representation of the hypergraph, where nodes are meetings and hyperedges (enclosing subsets of meeting nodes) represent the participants. Subfigure (b) shows a table of time interval edge attributes for each meeting: M1[0,1], M2[2,3], M3[1,4], and M4[3,5]. Subfigure (c) is a table detailing the topic incidence attributes, indicating which topics each person discussed in each meeting (e.g. P4 discussed {A, B} in M1 and {D} in M3). The figure collectively presents a hypergraph with temporal attributes on the meetings (edges) and topical attributes on the person-meeting incidences.

to 1). We note here that $Z = 2\mathcal{A}$. From these data we could directly try to tell which meetings influence which. To do this we need to account for both the temporal order of the meetings as well as the topics discussed at each meeting. Let us pose the question whether M_1 influenced M_2. We can see that M_1 did not share any of the same topics as discussed in M_2, but they did share person P_5 and M_1 did occur before M_2. The permissible walk graph encapsulates all of this information, as we will now demonstrate.

In this toy example, we demonstrate edge marginalization. Given the above incidence attribute function γ in Fig. 4c, we get an induced edge attribute function $g_Y : E \to Z$ defined as $g_Y(e) = \bigcup_{v \in R_E(e)} \gamma(v, e)$. That is, g_Y sends an edge $M_i \in E$ to the union of topic sets. Enumerating all evaluations of g_Y on the edges, we get $g_Y(M_1) = \{A, B, C\}$, $g_Y(M_2) = \{E, F\}$, $g_Y(M_3) = \{B, C, D\}$, and $g_Y(M_4) = \{C\}$. We apply this marginalization since, for example, it may be more useful to know the set of topics discussed in a meeting instead of which topics were discussed by which people. We also have another example of an edge marginalization function in the results of Section 4 for temporal attributes by taking the convex hull of intervals attributed to incidences.

We now construct the permissible walk graph from the attributed hypergraph in our toy example where we used an edge marginalization g_Y. The hyperedges are attributed with both the temporal attributes given by the original edge attribute function ε (with values in the set of time intervals) and the set attributes given by the function g_Y (with values in the subsets of all topics) induced from the original incidence attribute function γ.

We begin by calculating the attributed s-line graph $\mathcal{L}_s$ for $s = 1$ of the example hypergraph with only the vertex attribute function $\tau = \varepsilon$ being used. For a vertex $e \in E$ of $\mathcal{L}_s$, we have two functions that operate on the nodes: $\varepsilon(e)$ and $g_Y(e)$, where $\varepsilon(e)$ captures the interval attributes and $g_Y(e)$ the marginalized topic attributes. Figure 5a shows the attributed s-line graph. Note that each of these permissible graphs are directed specifically in virtue of the corresponding predicate q.

With the attributed s-line graph we now construct various permissible walk graphs using either or both of the node attributes. To do this we need to define the predicate functions. The first predicate we use for the interval attributes $i : \mathcal{I} \times \mathcal{I} \to \{0, 1\}$, where $\mathcal{I}$ is the set of intervals on $\mathbb{R}$, is based on the strong interval order [22]:

$$i([a, b], [c, d]) = \begin{cases} 1, & \text{if } b \leqslant c \\ 0, & \text{if } b > c, \end{cases} \tag{3.1}$$

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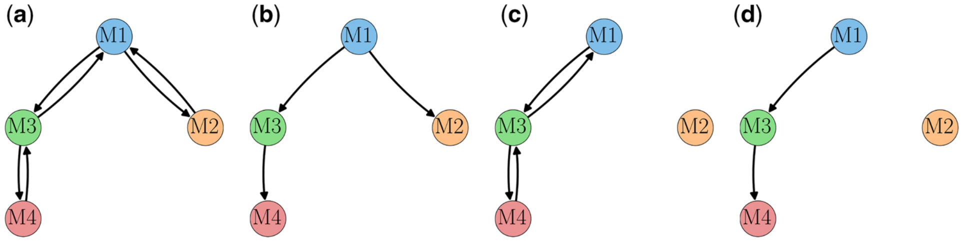

$\mathcal{L}_s(H)$. P_i. P_t. $P_{it} = P_i \cap P_t$.

Figure 5. (a) The attributed s-line graph $\mathcal{L}_s$ for $s = 1$, (b) interval order permissible graph P_i, (c) topic respecting permissible walk graph P_t, and (d) interval order and topic respecting permissible walk graph P_{it} for the toy example hypergraph in Fig. 4a.
Alt text: The figure displays four subplots (a-d) illustrating different graph representations derived from the toy example attributed hypergraph of meetings. All graphs have meetings (M1, M2, M3, M4) as nodes. Subfigure (a) shows the attributed s-line graph Ls(H) for s = 1, where nodes are meetings and undirected edges connect meetings sharing at least one participant. Subfigure (b) depicts the interval order permissible graph Pi, where directed edges indicate a strong temporal order between meetings (one ends before the other begins). Subfigure (c) shows the topic respecting permissible walk graph Pt, with undirected edges connecting meetings that discussed at least one common topic. Subfigure (d) illustrates the intersection of the previous two, Pit=Pi ⋂ Pt, capturing both the temporal order and shared topics between meetings with directed edges. The caption explains that (a) is the attributed s-line graph for s = 1, (b) is the interval order permissible graph, (c) is the topic respecting permissible walk graph, and (d) is the intersection of (b) and (c), representing both temporal order and shared topics for the toy example hypergraph.

where $[a, b], [b, c] \in \mathcal{I}$. This function captures one sense of the temporal directionality of the meetings, where if one meeting ends before another starts, there is a flow from that meeting to the next. We construct the corresponding permissible walk graph P_i by including only the edges $Q = \{(e_i, e_j) \in K\colon i(\varepsilon(e_i), \varepsilon(e_j)) = 1\}$ with ε as the edge attribute function of the attributed hypergraph, see Fig. 5b.

The second function $t : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$ only accounts for the topic sets on the vertices and determines if the sets intersect as

$$t(S_1, S_2) = \begin{cases} 1, & \text{if } S_1 \cap S_2 \neq \varnothing \\ 0, & \text{else,} \end{cases} \tag{3.2}$$

where S_1 and S_2 are sets. This functions checks to see which meetings have related topics, but has bidirectional edges as $t(S_1, S_2) = t(S_2, S_1)$. We construct the permissible walk graph P_t by including only the edges $Q = \{(e_i, e_j) \in K\colon t(g_Y(e_i), g_Y(e_j)) = 1\}$, see Fig. 5c.

The last permissible walk graph $P_{it} = P_i \cap P_t$ captures both the temporal directionality of the meetings as well as how the topics are related by taking the intersection of the other two permissible walk graphs. P_{it} is shown in Fig. 5d. With P_{it} we are able to now answer questions such as "did meeting M_1 influence the discussion in meeting M_2?" The permissible walk graph indicates that there is no observable mutual influence between M_1 and M_2 since there is no edge between the two. However, we can observe that M_1 could have influenced M_3 and M_4 as there are edges from M_1 to either and yet neither influenced M_1 due to the temporal directionality from i.

The computational cost of constructing the permissible walk graph is tied to the underlying s-line graph. A fully connected s-line graph, with all hyperedges intersecting, has $O(|E|^2)$ edges, where $|E|$ is the number of hyperedges. Constructing the attributed s-line graph depends on the attribute function complexities. The permissible walk graph is derived by filtering edges based on a predicate function q. In the worst case, q is evaluated for every potential edge. If q has complexity

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$O(f(n))$, where n is the input attribute size, the permissible walk graph construction is $O(|E|^2 \cdot f(n))$. However, hyperedges rarely fully intersect and q is often computationally inexpensive (e.g. interval comparisons). Therefore, the actual run-time is often significantly lower and comparable to the s-line graph construction itself.

4. APPLICATION: SOCIAL NETWORK ANALYSIS

In this section we consider the application of permissible walks to the Reddit social media network, where the vertices are users and the hyperedges represent discussion threads in which the users post messages. These structures are temporally attributed in that there are time stamps on messages, supporting time intervals on users, threads, and users' activity in threads, in terms of the initial and final posting times.

If we wanted to see the flow of traffic through posts on particular "subreddits" (topic boards within Reddit as a whole) we could use s-line graphs to recover which posts share a certain number of users. But this would fail to capture the direction of the flow over time; i.e. users going from one post to another. Our first analysis studies the intra-subreddit permissible walks by demonstrating how the permissible walk graph can capture the influence of a thread on other threads. Following this we study the inter-subreddit relations by showing the interaction matrix between subreddits as a summary of the permissible walk graph, which captures information about connectivity and directionality of communication between subreddits. Additionally, throughout this analysis we make comparisons to the standard s-line graph approach that could be used to study the interactions between subreddits.

4.1 Data

The Reddit dataset is from the PAPERCRANE Social Network Problem-shop [23]. The dataset consists of COVID-19 related subreddits for 320 days from January 15 to November 30, 2020. The data is of the author posts, which includes information on the time of post, thread ID, and subreddit.

Over the duration of the study time, n users have posted at least once to at least one of m different threads. As each post is time-stamped, there is an $n \times m$ "post array" W whose cells $W[i][j]$, $i \in [n]$, $j \in [m]$ contain a list of the time stamps for the posts that user i made to thread j. For each such cell $W[i][j]$, we can then generate a time interval $I_{i,j}$ simply as the min and max extrema:

$$I_{i,j} := [\min W[i][j], \max W[i][j]] \, .$$

Given this array of intervals, we can derive intervals which describe the time frame in which a thread was active, and similarly when a user was active. We define I_i^{row} and I_j^{col} to be the convex hull over intervals in the ith row and jth column respectively. Specifically, let $\uplus$ denote the convex hull, giving us:

$$I_i^{row} = \biguplus_{j=1}^{m} I_{i,j} = \left[\min_{j=1}^{m} I_{i,j}, \max_{j=1}^{m} I_{i,j} \right], \qquad I_j^{col} = \biguplus_{i=1}^{n} I_{i,j} = \left[\min_{i=1}^{n} I_{i,j}, \max_{i=1}^{n} I_{i,j} \right].$$

We note that I_i^{row} is a row marginalization on users, and I_j^{col} is a column marginalization on threads. This is cartooned in Table 1, showing the user and thread time intervals on the margins of the post array. In the example, $I_{n,1} = [5, 10]$, while $I_{j=m}^{col} = [13, 13] \uplus [2, 7] = [2, 13]$. Note that user 1 did not post to thread m, nor did users 2 or n post to thread 2.

The dataset includes 12 subreddits which are listed in Fig. 6. This figure also helps visualize the size of each subreddit over the 10 months of data by showing the trace of intervals associated to the threads.

Definition 4.1 Assume there exists a collection of intervals $\mathcal{I} = \{I_i\}_{i=1}^{n}$ over support $U(\mathcal{I}) = \biguplus_{i=1}^{n} I_i$, then their **trace** [24] is a function $T \colon U(\mathcal{I}) \to \mathbb{Z}_{\geqslant 0}$ which counts the

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Table 1. The post array A with a list of post times in the cells and marginal time intervals on users and threads.

		$[1, 10]$ Thread 1	$[4, 7]$ Thread 2	$\cdots$	$[2, 13]$ Thread m
$[1, 7]$	User 1	$1, 2, 3$	$4, 7$	$\cdots$	$\emptyset$
$[8, 13]$	User 2	8	$\emptyset$	$\cdots$	13
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$[2, 10]$	User n	$5, 10$	$\emptyset$	$\cdots$	$2, 7$

The figure is a table showing a post array A for Reddit data, with post timestamps in cells for users across threads. Marginal time intervals for each user (I row i) are shown on the left, representing their overall activity period. Marginal time intervals for each thread (I col j) are displayed at the top, indicating the thread's active period. For example, User 1 posted in Thread 1 at times 1, 2, 3 and in Thread 2 at 4, 7, with an overall activity from 1 to 7.

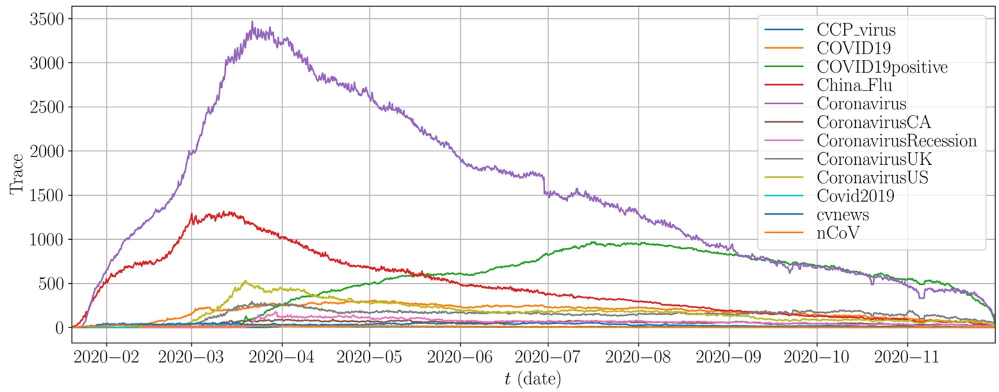

Figure 6. Trace $T(t)$ (interval count at time t) of the collection of temporal intervals I_j associated to each thread for COVID-19 related subreddits from the PAPERCRANE [23] dataset.
Alt text: The figure displays a trace plot showing the number of concurrently active Reddit threads over time $(\mathrm{T(t)})$ for various COVID-19 related subreddits from the PAPERCRANE dataset. The x-axis represents time, spanning the duration of the data collection, and the y-axis represents the count of active threads. Each colored line corresponds to a different subreddit, illustrating how the number of active threads within that subreddit fluctuated over the observed period. The plot provides a visual summary of the temporal dynamics and activity levels within each of the studied subreddits.

number of active intervals at any time $t \in U(\mathcal{I})$, so that

$$T(t) = |\{I_i \in \mathcal{I} | \min(I_i) \leqslant t \leqslant \max(I_i)\}|.$$

Figure 6 shows the traces of the thread intervals I_j for the subreddits, with t limited to 2000 evenly spaced instances spanning the support of the interval collection.

4.2 Intra-subreddit analysis

The first motivation for studying the PAPERCRANE dataset using permissible walks is to gain some understanding of the intra-subreddit communication structure: which threads within a subreddit influence other threads? Specifically, We are interested in how users participate in online discussions. Our framework shows temporal links between threads where the same users are active, but we do not see these links as direct cause and effect. Users might join different threads for many reasons—external events, platform algorithms, or simply shared interests. We aim to describe these participation patterns and how they change, giving us a picture of user interaction in online spaces. Shared users suggest something about the community: its connections, evolution, and information

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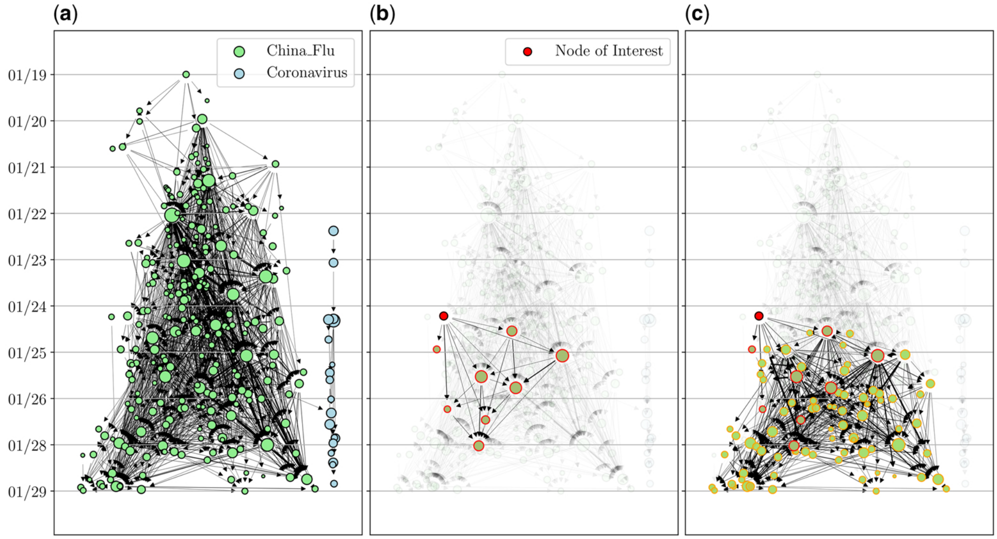

Figure 7. Intra-subreddit analysis for Coronavirus and China_Flu subreddits using the permissible walk graph (a) as well as the directly connected (neighbor) downstream nodes of a node of interest (b) and all downstream nodes of the node of interest (c).
Alt text: The figure presents an intra-subreddit analysis using a permissible walk graph (s=10, strong-order predicate) for the Coronavirus (orange) and China_Flu (blue) subreddits, visualizing temporal flow of user participation between threads (node size = author count, edge width = shared author count). Subfigure (a) shows two largely separate subreddit communities with few connections. Subfigure (b) highlights a "Node of Interest" and its eight immediate downstream neighbors (temporally subsequent threads with at least ten shared authors). Subfigure (c) illustrates all downstream nodes from this node, showing the potential reach of user engagement originating from it. This directional analysis reveals potential influence within and between subreddits over time, unlike a non-directional s-line graph.

flow. We focus on what users do, not necessarily why. This helps us to understand the dynamics of the online community by looking at observable participation patterns. More research could incorporate more data to explore the different factors that influence user behavior.

To build this intra-subreddit analysis, we will use an example from the China_Flu and Coronavirus subreddits in January 2020. During these first two weeks of data, both subreddits have grown in popularity, as shown in Fig. 6. Our goal is to see how the permissible walk graphs evolve during these two weeks and the spread of downstream connections of an example thread on all the other threads.

Figure 7 shows the color coded permissible walk graph for the China_Flu and Coronavirus subreddits. This graph was constructed with $s = 10$ to allow visualization as the graph is too large to practically display for lower s. Additionally, we used the strong-order predicate as described in Section 2 (3.1). For visualization, we have also removed isolated nodes (85 for this example) in the graph. Additionally, node sizes and edge widths are correlated to the number of authors in a thread and the number of intersecting authors between threads, respectively.

We see that initially, China_Flu is the dominant subreddit. However, around January 22 Coronavirus begins to have threads with size $\geqslant 10$. Additionally, the graph for this s shows that there are clearly two main communities that are separated by subreddit assignment with only a total of 6 edges connecting the two.

To understand the dynamic relationship of a thread to all downstream threads, we used a demonstrative thread of interest (Node of Interest in Fig. 7b). We can see that this node of interest has only 8 immediately downstream neighbors, meaning these threads shared at least 10 co-authors

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and occurred temporally after the node of interest. This visualization helps understand the direct connections between this thread and others based on shared user participation and temporal order. Figure 7c shows all downstream nodes from the thread of interest, illustrating the potential reach of user participation originating from this node for s = 10. This information can be valuable when considering the potential social effects of the discussion within the node of interest.

A similar analysis would not be possible with just the s-line graph, as it lacks directionality information. The s-line graph would treat all connected nodes as equally related to the node of interest, regardless of temporal order. This would incorrectly suggest a relationship even when threads ended before the node of interest began, meaning it cannot accurately trace the flow of user participation.

4.3 Inter-subreddit analysis

The previous analysis can also be done for inter-subreddit communications for understanding the exact threads of interaction between subreddits. To do this, we utilize the interaction matrix M. For our purposes, we will consider each subreddit a "class."

Definition 4.2 Let $\mathcal{C} = \{c_1, \ldots, c_r\}$ be a set of classes, and let G be a node attributed directed graph with class type as an attribute, i.e. we have a vertex attribute function $C : V(G) \to \mathcal{C}$. We define the **interaction matrix** M to be the $r \times r$ matrix where $M_{i,j}$ is given by the number of edges $(s, t) \in E(G)$ such that $C(s) = c_i$ and $C(t) = c_j$.

We can think of the interaction matrix as the adjacency matrix of a weighted, directed graph (which may contain directed loops) on the node set $\mathcal{C}$ which describes the interactions between classes in G, which we will call the **class graph**. For the Reddit data, our classes are the subreddits which the given threads are a part of, so the class graph has subreddits as nodes and has weighted, directed edges based on the number of edges in the permissible walk graph from one subreddit to another. It should be noted that the interaction matrix does not capture information about how many components there are within a subreddit, just the number of connecting edges between subreddits. However, as is shown in Fig. 8, we include the number of components and show that the subreddits tend to predominantly be composed on a single component. This formation captures the inter-subreddit communication. If the resulting interaction matrix is zero off-diagonal, then there is no interaction between subreddits.

We now demonstrate how the interaction matrix captures information about both connectivity of subreddits as well as directionality in the flow of communication between the subreddits of COVID19 and China_Flu. For these subreddits there is a high asymmetry in the interaction matrix. As shown in Fig. 6, the subreddits China_Flu and COVID19 begin at different times. This is due to COVID-19 not yet being official named. Because of this, alternate names such as China_Flu were used. On February 11 the World Health Organization announced COVID-19 as the official name of coronavirus and consequently, the COVID19 subreddit quickly grew in popularity. We theorize this corresponded to a shift of authors from China_Flu to COVID19. This possible influx of authors from China_Flu to COVID19 is captured using the interaction matrix as shown in Fig. 8 for $s = 1, 10, 20$.

At $s = 1$, there are approximately 220k edges from China_Flu to COVID19 and only 42k edges from COVID19 to China_Flu. This shows the asymmetry of authors originally posting in China_Flu moving to COVID19. Additionally, there are only 29k self edges for COVID19 at $s = 1$, which is relatively low compared to the number of edges connecting to China_Flu. However, at $s = 10$, this switches from with 1.7k self edges for COVID19, and only a total of 18 edges connecting COVID19 to China_Flu. This shows that the edges that are connecting the two subreddits do not represent a large number of authors in the intersection of threads. This trend is more clearly shown in the bottom subfigure of Fig. 8, where the number of cross subreddit edges is decreasing at a much higher rate than the self edges. By increasing s and doing this analysis we are able to show that "strength" of the connectivity between subreddits and that the threads

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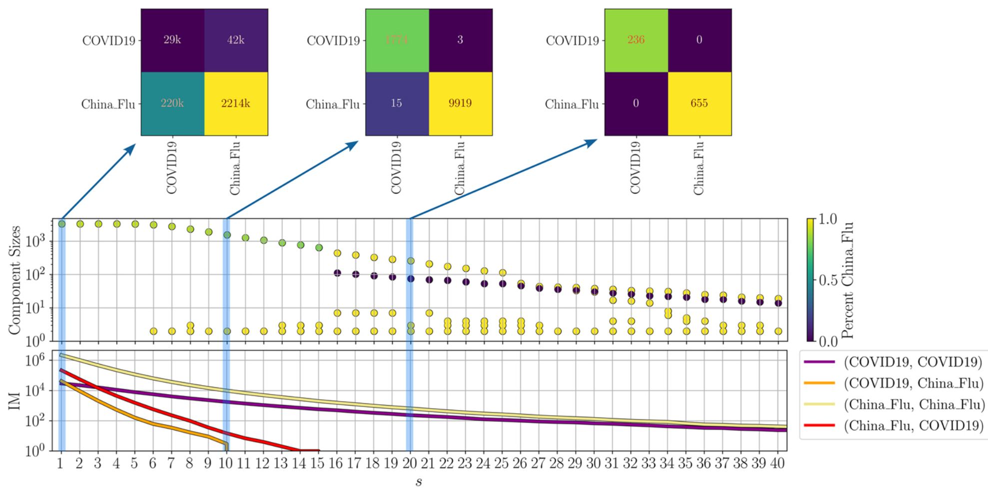

Figure 8. Interaction matrices at various *s*-values demonstrating connectivity of edges between subreddits COVID19 and China_Flu in the permissible walk graph. Additionally, the number of components and their relative size are provided. In the top chart, each point represents the number of nodes in a weakly-connected component in the permissible walk graph. In the bottom figure are the values of the interaction matrices as *s* increases showing the changing connectivity between the two subreddits in the permissible walk graph.
Alt text: The figure presents two linked visualizations analyzing the connectivity between the COVID19 and China_Flu subreddits using interaction matrices derived from permissible walk graphs at varying s values. The top chart shows the size distribution of weakly-connected components in these graphs, illustrating how the network fragments as s increases. The bottom part displays 2×2 interaction matrices for s=1,10,20, quantifying the directed edges between threads of the two subreddits. The matrices reveal a strong initial flow of connections from China_Flu to COVID19 at s=1, which diminishes as s increases, indicating that cross-subreddit connections are based on smaller author overlaps compared to within-subreddit connections. This analysis demonstrates how the strength and directionality of inter-subreddit communication, based on temporally ordered threads with shared authors, changes with the stringency of the overlap requirement s.

within a subreddit share more common authors on average per intersection of threads than do cross subreddit threads. Starting at $s = 15$ and shown for $s = 20$, we see a bifurcation in the number of components going from one main component to two for each of the subreddits (color coded with yellow as China_Flu and Purple as COVID19). We also note that there are many small components beginning at $s = 6$.

5. CONCLUSION

In this work we developed a framework for studying attributed hypergraphs using the permissible walk graph. The permissible walk graph is a generalization of the commonly used *s*-line graph by accounting for hypergraph attributes when creating a directed version of the *s*-line graph that respects the desired relation between attributes. The main application we focused on in this work was temporal attributes and how they can be used to improve the analysis of social network data in comparison to the *s*-line graph. However, the permissible walk graph can be used to study many different attribute types that may exist on hypergraphs such as types, hierarchical, sets, and Boolean attributes.

Our results showed that the permissible walk graph was able to capture the spread of communication between threads in multiple COVID-19 related subreddits from the PAPERCRANE reddit dataset [23]. Additionally, it was able capture the dynamics of authors posting in one subreddit to

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another by using an interaction matrix as a summary of the permissible walk graph, which is not captured when using the *s*-line graph. Lastly, we showed how the permissible walk graph at different values of *s* can capture the strength of communication between subreddits with edges connecting subreddits sharing smaller intersections of authors than those within a single subreddit. More generally we hope that this will serve as an example driven explanation of how to use our framework for tracking potential spread of information or disease in attributed higher order networks.

Future research directions using this tool include using the permissible walk graph for clustering and implementation of graph neural networks for classification. Additionally, we plan to develop new relations for directionality information of ontological attributes to help in the study of knowledge hypergraphs. On the more theoretical side one question that still remains is how to carefully handle marginalization with different attribution types.

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