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Equivalence of Hypergraph Categories 2nd Edition

August 2024

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U.S. DEPARTMENT OF ENERGY

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Contents

1 Introduction

This is an expository paper working through the details of some categorical equivalences, and non-equivalences, between a-priori different, but closely related, notions of a hypergraph category. The paper is intended to be self-contained, and is written for an audience not familiar with category theory, but potentially interested in category theory as a tool for processing of hypergraphs in a computational pipeline. As such, it "turns the crank" on some details usually left to the reader by literature. Some of the arguments are at times slightly non-standard, and from a mathematical perspective, even aesthetically unpleasing, in order to make a closer connection to the way information is represented in an implemented system and to reduce reliance on familiarity with categorical concepts. Most of these categories are just the category of Dorfler-Waller[2] with different representational choices. At the time this work was written we were unaware of the incidence-based approach of Grilliette and Rusnak [3], which has some advantages from a category theory perspective and is only touched on here in Section 6.2.

Roughly, a category consists of objects, along with structure-preserving transformations, called morphisms. The basic idea behind category theory is that the transformations between objects, the morphisms, are the meaningful structure the category, not the structure defined on the objects themselves. Category theory allows us to formalize a notion that two categories are "not different in any way that matters", or in the parlance, categorically equivalent. More generally, category supports the formal notion of a functor, which maps structure in one category to another. Category theory allows us to understand how and when changes in the way objects are represented and/or transformed will lead to meaningful changes in mathematical structure, allowing us to ignore, or as desired, litigate, changes in representational structure. [1]

2 BLUF

The following five categories of hypergraphs are equivalent:

1. The category $\mathcal{HG}$ of hypergraphs as triplets (V, E, I) (Example 5.3),
2. The category $\mathcal{OHG}$ of hypergraphs as triplets (V, E, I), where V and E are ordered and I is a matrix (Example 5.5),
3. The category $\mathcal{HG}_{BINREL}$ of hypergraphs which are binary relations (Example 6.4),
4. The category $\mathcal{HG}_{BMAT}$ of hypergraphs which are boolean matrices (Example 6.2),
5. The category $\mathcal{BIC}$ of hypergraphs as bicolored graphs. (see Definition 6.13).

We provide definitions and arguments of the above statements. We also introduce some general category theory notions and language, intended to be helpful to the novice and for review but not a substitute for a complete treatment.

We discuss some issues involving slight variants of the definitions which in some cases do and in some cases do not affect the nature of our category. For example, distinctness requirements between edges and vertices do not affect any categorical structure (Proposition 6.7), but the choice of how morphisms are defined does (Proposition 6.8).

3 Hypergraph Definitions

In this section, for simplicity, we assume all sets are finite.

Definition 3.1. *A* hypergraph H *is a triple* (V, E, I) *consisting of*

- *A set of* vertices V*,*
- *A set of* edges E*, and*
- *An* incidence function $I : V \times E \to \{0, 1\}$*.*

We say $v \in e$ *if* $I(v, e) = 1$*.*

The dual hypergraph H^* *of* H *is the hypergraph* (E, V, I^*)*, where* $I^* : E \times V \to \{0, 1\}$ *is defined such that* $I^*(e, v) = I(v, e)$*.*

Notes and Variants:

1. One may take V and E to be totally ordered sets, and define I using a matrix.

[1] Special thanks to William Grilliette, who read an early draft of this work and suggested corrections to proofs.

- A (finite) *totally ordered set* is an ordered pair (X, f), where X is a (finite) set and $f : \{0, \ldots, |X| - 1\} \to X$ is a bijection.
- Some would argue that in Definition 3.1, I is already a matrix; it just has rows and columns indexed by (unordered) sets rather than by natural numbers. When working with graphs, it is common to speak informally about an "adjacency matrix" even when no order on the vertices has been specified. Below, the distinction is made explicitly to help sort out these issues.
- Sometimes E is additionally taken to be an indexed family of subsets of V, that is $e_j = \{v \in V | I(v, e_j) = 1\}$. In this case the structure of the incidence function and matrix I can be constructed from E as $I|_{V \times e} = \chi_e$, where χ_e is the characteristic function on the subset e of V.
- Similarly, one can define a function $f : E \to \mathcal{P}(V)$ that maps edges to subsets.

2. [1] uses totally ordered sets, with edges as subsets. The choice whether or not to order vertices and edges is inessential unless the orders have some particular use. Defining incidences rather than edges makes the dual a bit easier to work with. In particular, the double dual of a hypergraph H is H when we define it by incidences, whereas under the hyperedge-defined variant the most natural way to define the dual makes H^{**} only naturally isomorphic to H. See [1] for the hyperedge-based definition of the dual. This (category theory) argument will be made precise later.

3. This definition does not require that V and E be disjoint. It is neither necessary, nor does it hurt to assume, that vertex and edge sets are disjoint because we decide the roles of objects before computing incidences; one may replace V and E with $V \times \{0\}$ and $E \times \{1\}$ respectively if disjoint sets are needed. The category of hypergraphs is equivalent to the (sub)category of hypergraphs with disjoint vertex and edge sets; see Proposition 6.7. Overloading the meaning of "$v \in e$" to capture incidence does admittedly present some confusion since the phrase can be interpreted in two ways, with different resulting meanings. When there is danger of confusion, the language "v is a vertex of edge e" removes the ambiguity.

4. In [1], vertices and edges can be equal (depending on what the undefined term "elements" means) but edges are specified by index within the indexed family, which disambiguates the intended role.

Some definitions referred to in [1] do not allow duplicate hyperedges. To talk about such hypergraphs, one option is to copy the terminology for graphs:

Definition 3.2. *A* simple hypergraph *is a hypergraph such that for any* $e_1, e_2 \in E$*, the functions* $I(\cdot, e_1), I(\cdot, e_2) : V \to \{0, 1\}$ *are equal only when* $e_1 = e_2$*.*

Discussion: The term *graph* is often used somewhat loosely. A *simple graph*:

- is not a multigraph: multiple edges with the same vertex sets are not allowed,
- does not admit edges which connect a vertex to itself.

It is reasonable to use *multihypergraph*, or *vertex-multi hypergraph* to explicitly indicate that multiple edges can have the same vertices, *dual-multi hypergraph*, or *edge-multi hypergraph* to indicate that multiple vertices can have the same containing edges, and *multi, dual-multi hypergraph* or *edge and vertex multihypergraph* (or just, *hypergraph*) to indicate both.

Note that a simple hypergraph need not have a simple dual hypergraph, so one may need to say "dual simple", "simple, dual simple", "vertex simple", etc. as appropriate.

We will mostly not be concerned with self loops, but for completeness, here is one way to define them for hypergraphs:

Definition 3.3. *A* hypergraph with self-loops *is a triple* (V, E, I)*, with* V *a* vertex set*,* E *an* edge set*, and* $I : V \times E \to \mathbb{N}$*.*

The hyperedge-based variant of this incidence-based definition replaces the sets that comprise edges E with multisets. Along with this change, the definition of the dual in [1] needs to be altered to account for multiplicity (in a way that makes it compatible with our I^*), or else the double dual won't be isomorphic to the original hypergraph.[2] Unlike the property of being simple, if a hypergraph has self-loops, so does its dual, as there are multiple incidences between the same vertex and edge.

Duplicate incidences can be represented by extending the domain of the incidence function from $\{0, 1\}$ to $\mathbb{N}$, if only the number of incidences is needed, or by an indexing set if it is necessary to define properties (functions) on the individual incidences.

4 Some related structures

In this section, assume all sets are finite.

Definition 4.1. *A* binary relation R *on sets* X *and* Y *is a subset* $R \subset X \times Y$ *with the notation* $R(x, y)$ *indicating that* $(x, y) \in R$*. Equivalently, one may define* R *via its characteristic function* $\chi_R : X \times Y \to \{0, 1\}$*, with* $\chi_R(x, y) = 1$ *iff* $(x, y) \in R$*.*

Example 4.2. *The triple* (X, Y, χ_R) *is a hypergraph.*

Definition 4.3. *An* $m \times n$ *boolean matrix M is a function* $M : \{1 \ldots m\} \times \{1 \ldots n\} \to \{0, 1\}$*. The transpose* M^T *of* M *is the function* $M^T : \{1 \ldots n\} \times \{1 \ldots m\} \to \{0, 1\}$ *such that* $M^T(a, b) = M(b, a)$*.*

Example 4.4. *The triple* $(\{1 \ldots m\}, \{1 \ldots n\}, M)$ *is a hypergraph.*

Definition 4.5. *Let* $G = (V, E)$ *be a directed or undirected graph or multigraph. A* bipartition *on* G *is a pair of subsets* $V_1 \subset V$*,* $V_2 \subset V$*, such that:*

1. $V_1 \cap V_2 = \emptyset$*,*

[2] For non-experts, taking the dual is analogous to swapping the roles of edges and vertices in a hypergraph; we want this notion to be meaningful, as well as involutive: the dual of a hypergraph should be a hypergraph, and the dual of the dual should be the original hypergraph, possibly up to some structurally insignificant representational choices, or more precisely, up to categorical equivalence.

2. $V_1 \cup V_2 = V$,

3. *For every edge* $(v_1, v_2) \in E$ *(in the simple graph case:* $\{v_1, v_2\} \in E$*),* $v_1 \in V_1$ *iff* $v_2 \in V_2$.

A bipartite (multi)graph *is a graph which admits a bipartition. A* bipartitioned (multi)graph *is a bipartite graph with a choice of bipartition.*

Notes:

1. Non-empty graphs with self-loops cannot admit bipartitions.

2. Bipartitions are in one-to-one correspondence with bicolorings, introduced in [5]. In particular, a bicoloring is the characteristic function of one of the subsets V_1, V_2 in the bipartition.

Example 4.6. *The graph* G *has an adjacency function* $A^G : V \times V \to \{0, 1\}$.
If G *has a bipartition into vertices* $V = V_1 \sqcup V_2$*, the triple* $(V_1, V_2, A^G|_{V_1 \times V_2})$ *is a hypergraph.*

Generally one intends here that G is undirected. If G is directed, we can define three hypergraphs, in general non-isomorphic, and the duals of these, using edges from V_1 to V_2, edges from V_2 to V_1, or both:

- $(V_1, V_2, A^G|_{V_1 \times V_2})$,
- $(V_1, V_2, (A^G)^T|_{V_1 \times V_2})$,
- $(V_1, V_2, A^{\bar{G}}|_{V_1 \times V_2})$, where $\bar{G}$ is the minimal graph containing G such that $A^{\bar{G}}$ is a symmetric function.

The above hypergraph construction is well-defined (and if G is directed, all three constructions are well-defined) even when G is not bipartite. All that is required is a pair of vertex sets V_1, V_2.

Example 4.7. *The* neighbor hypergraph (V, V, A_G) *is a hypergraph with the same vertices as* G*, and a hyperedge for each maximal collection of vertices that share a particular common neighbor. The identity of that neighbor distinguishes multiple, otherwise identical hyperedges.*

5 Categories and equivalences

5.1 Categories

The purpose of this section is to lay out the definitions and propositions required to prove the equivalences in the next section. The propositions stated are all standard; proofs may be found in an introductory category theory text such as [4].

Consider the following variant of Definition 3.1, the definition of a hypergraph:

Definition 5.1 (definition of a totally ordered hypergraph)**.** *A* vertex-and-edge-totally-ordered hypergraph H *is a triple* (V, E, I) *consisting of*

- *A totally ordered set of* vertices $V = \{v_1, \ldots, v_m\}$,

- *A totally ordered set of* edges $E = \{e_1, \ldots, e_n\}$*, and*
- *An* $m \times n$ *matrix I,* called the incidence matrix*, taking coefficient values in* $\{0, 1\}$*.*

We say $v_i \in e_j$ *if* $I_{i,j} = 1$*.*

The dual vertex-and-edge-totally-ordered hypergraph H^* *of* H *is the hypergraph* (E, V, I^T)*.*

Category theory addresses the question of whether differences between objects are essential in the following way: a collection of objects is associated with a collection of transformations (of some sort) between those objects. The behavior of these objects is given by the transformations we may perform on them. If a transformation is reversible, the objects are not different in structure.

This is the formal definition of a category:

Definition 5.2 (Definition of a category)**.** *A* category $\mathcal{C}$ *is a triple* $\mathcal{C} = (\mathcal{C}^0, \mathcal{C}^1, \circ)$*, consisting of:*

1. *A class of* objects $\mathcal{C}^0$*,*
2. *A class function* $\mathcal{C}^1$ *assigning each* $a, b \in \mathcal{C}^0$ *to a set* $\mathcal{C}^1_{a,b}$*, called the set of* morphisms from a to b*, or* arrows from a to b*,*
3. *A class function* $\circ$ *assigning to each triple* (a, b, c) *of elements of* $\mathcal{C}^0$*, a set function* $\circ_{a,b,c} : \mathcal{C}^1_{a,b} \times \mathcal{C}^1_{b,c} \to \mathcal{C}^1_{a,c}$*, which is*
   (a) *associative:* $(f \circ g) \circ h = f \circ (g \circ h)$*, and*
   (b) *has identity: for each* $a \in \mathcal{C}^0$ *there exists* $i_a \in \mathcal{C}^1_{a,a}$ *such that for all* $b \in \mathcal{C}^0$ *and all* $f \in \mathcal{C}^1_{a,b}$*,* $g \in \mathcal{C}^1_{b,a}$*,* $i_a \circ f = f$ *and* $g \circ i_b = g$*.*

Small points:

1. Care is required in mathematics to avoid paradoxes when collections of objects are defined notionally, for example, when we wish to treate the collection of all graphs as an object. The use of the term “class”, as distinct from “set”, is necessary to avoid paradoxes in these situations. For those doing applied work, the necessity is generally present only when laying out definitions and notions; the distinction between classes and sets often plays no role in the work that is eventually done.
2. We use a composition convention which is different from the *composition of functions* convention used many places in mathematics. Our convention is called *composition of arrows*. Composition of arrows preserves the left-to-right order of transformations for left-to-right readers. The choice is a minor convenience or a minor inconvenience depending on context..
3. i_a is unique: if i'_a is an identity morphism for a, then i_a = $i_a i'_a = i'_a$.
4. Indices are often dropped when they can be safely inferred from context.

Example 5.3. *The category* $\mathcal{HG}$ *of hypergraphs (Definition 3.1) with hypergraph morphisms. See the definition immediately below.*

Definition 5.4. *Let $H_1 = (V_1, E_1, I_1)$, $H_2 = (V_2, E_2, I_2)$ be hypergraphs.*

Given $e \in E_1$, let $f_V(e)$ denote $f_V(\{v \in V_1 | v \in e\})$ (recall that $v \in e$ has nonstandard meaning defined via the incidence function), and for $e \in E_2$ $V(e)$ denote the set $\{v \in V_2 | I_2(v, e)\}$.

A hypergraph morphism *$f : H_1 \to H_2$ is a pair of functions $f_V : V_1 \to V_2$ and $f_E : E_1 \to E_2$ such that for each $e \in E_1$, $f_V(e) = V(f_E(e))$. In other words, f_V maps the vertices of a hyperedge e surjectively to the vertices of its image hyperedge $f_E(e)$.*

Example 5.5. *The category $\mathcal{OHG}$ of totally ordered hypergraphs, with hypergraph morphisms.*

One may well ask, shouldn't we have required that the morphisms in this category respect the orders on the vertices and edges? The answer is that this is only necessary if the orders on the vertices and edges are structure that we wish to preserve. It turns out that order preservation is not a very natural property for hypergraph maps; we know of no application for hypergraph maps which preserve vertex and/or edge order, beyond the trivial but occasionally useful observation that constructing an isomorphism between hypergraphs is a far easier problem when the vertex or edge order is fixed.

As a follow-up, one may ask, if the order isn't preserved, why did we include it in the definition at all? The example is pedagogical, and is intended for those working within an information processing pipeline, who may not be free to jettison currently-inessential structure which may be needed later.

Category theory's focus on morphisms as the carriers of object structure frees us somewhat from the need to be fussy about the exact representational choices made within our objects. We can choose what is and isn't important by our choice of morphisms.

An example of a vertex order which provides inessential structure that is nonetheless worth keeping is provided by set-system hypergraphs which are connected to a simplicial complex construction. The vertices in a simplicial complex are typically ordered, in order to construct oriented simplicies and thus a homology functor, but the choice of order only affects the homology functor up to a choice of basis in the codomain space. Given a simplicial map between two simplicial complexes, the orders on the vertices determine choices of basis for the homology map but they do do not determine whether or not a map is simplicial. The order is, from a structural standpoint, inessential, and therefore morphisms are not required to preserve it, but it is also needed for computations so we keep it as part of the object structure.

Example 5.6. *In Definition 5.4, we could replace the requirement $f_V(e) = V(f_E(e))$ with the following: $f_V(e) \subset V(f_E(e))$, giving a new category $\mathcal{HG}'$. In other words, the vertices $\{v_1, \ldots, v_n\}$ of a hyperedge e need to be mapped by f_V to vertices of $f_E(e)$, but not surjectively; $f_V(e)$ can be properly contained in vertices of $f_E(e)$. This is different than the requirement, considered in [5] that f_E be surjective.*

Note: $\mathcal{HG}'$ is either, depending on the notation being used, the category $\mathcal{REL}$ of relations and morphisms of relations, or the double part of the double category $\mathcal{REL}$ of sets and relations.

5.2 Isomorphisms

Definition 5.7. *Let $\mathcal{C}$ be a category, $a, b \in \mathcal{C}^0$. An* isomorphism *is a morphism $f : a \to b$ such that there exists a morphism $f^{-1} : b \to a$ such that $f \circ f^{-1} = i_a$, $f^{-1} \circ f = i_b$. f^{-1} is said to be the* two-sided inverse *for f. If such an f exists, a and b are said to be isomorphic objects.*

Isomorphic objects are not structurally different in any way; it turns out that any object property on a one can describe via arrows between objects also holds for the isomorphic object b by passing through the isomorphism.

Isomorphism is an equivalence relation on objects, and divides the objects of a category into equivalence classes,[3] called *isomorphism classes*.

Example 5.8. *A hypergraph isomorphism $f : (V_1, E_1, I_1) \to (V_2, E_2, I_2)$ is a hypergraph morphism in which both f_V and f_E are bijections. The inverse hypergraph morphism is (f_V^{-1}, f_E^{-1}).*

Example 5.9. *In example 5.6, brief consideration shows that if (f_V, f_E) is an isomorphism in $\mathcal{HG}'$, f_V and f_E are bijections. Since $f_V(e) \subset V(f_E(e))$ and $f_V^{-1}(V(f_E(e))) \subset V(f_E^{-1}(f_E(e))) = V(e)$, $V(e)$ and $V(f_E(e))$ must also have the same cardinality, so edges are mapped surjectively to edges of the same size. Thus any isomorphism in $\mathcal{HG}'$ is also an isomorophism in $\mathcal{HG}$. Conversely, an isomorphism in $\mathcal{HG}$ is also an isomorphism in $\mathcal{HG}'$. Thus $\mathcal{HG}$ and $\mathcal{HG}'$ have the same objects, and the same isomorphism classes of objects. They are, however, different categories, as shown in Proposition 6.8.*

5.3 Functors and natural transformations

Two issues frequently occur when we describe the "same" class of objects with two different formalisms:

1. Each description leads to isomorphism classes with many equivalent objects,
2. There isn't a good way (or any way) to match objects bijectively between the formalisms.

Above, a hypergraph (Definition 3.1) only becomes associated with a (particular) binary matrix when we choose an ordering on the vertices and edges. In Definition 3.1, we have no canonical way to assign a boolean matrix to a hypergraph, and in Definition 5.1, we do. But does the difference matter?

The answer, it turns out, is that it may or may not, depending on whether the morphisms we care about (the ones "in our category") are required to preserve order.

To discuss relations between categories, we must have a way to relate them. A functor is a "morphism between categories". It sends objects to objects, and arrows to arrows in a way that respects composition and identity.

Definition 5.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* functor *$F : \mathcal{C} \to \mathcal{D}$ is a pair (F^0, F^1) such that:*

1. *F^0 is a class function that assigns each $a \in \mathcal{C}^0$ to $F^0(a) \in \mathcal{D}^0$.*
2. *F^1 is a class function assigning each pair $a, b \in \mathcal{C}^0$ to a class function $F^1_{a,b} : \mathcal{C}^1_{a,b} \to$*

[3] This is the same not-defined-here use of "class" as above.

$\mathcal{C}^1_{F^0(a),F^0(b)}$.

3. *(Identity) For all* $a \in \mathcal{C}^0$, $F^1(i_a) = i_{F^1(a)}$.
4. *(Associativity) For all* $a, b, c \in \mathcal{C}^0, f : a \to b, g : b \to c$, $F^1(f \circ g) = F^1(f) \circ F^1(g)$.

Example 5.11. *For any category* $\mathcal{C}$*, the* identity functor $Id : \mathcal{C} \to \mathcal{C}$ *sends objects and morphisms to themselves.*

Definition 5.12. *Let* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{E}$ *be functors.*

The composition of functors $F \circ G$ *is the pair* $((F \circ G)^0, (F \circ G)^1)$*, where* $(F \circ G)^0(a) = F^0(G^0(a))$*, and* $(F \circ G)^1_{a,b}(f) = G^1_{F^0(a),F^0(b)}(F^1_{a,b}(f))$.

Proposition 5.13. $F \circ G$ *is a functor.*

Example 5.14. *The (quasi)category* $\mathcal{CAT}$ *of categories, with categories as objects and functors as morphisms.*

We have to say "quasicategory" because there are too many category object classes to form a class without creating a paradox.

Example 5.15. *For any category* $\mathcal{C}$*, for each object* a*, choose an isomorphic object* a' *and an isomorphism* $\eta_a : a \to a'$*. Then there is a functor* $F : \mathcal{C} \to \mathcal{C}$ *such that each* $F^0(a) = a'$ *and for each* $a, b \in \mathcal{C}^0$, $F^1_{a,b}(f) = \eta_a^{-1} \circ f \circ \eta_b$.

The previous example can be thought of as a deformation of the identity functor by isomorphisms on the objects.

Definition 5.16. *Let* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{D}$ *be functors. A* natural transformation η of functors from F to G *is a family of morphisms* $\eta_a : F^0(a) \to G^0(a)$*, one for each* $a \in \mathcal{C}^0$*, such that for each* $a, b \in \mathcal{C}^0$ *and each* $f \in \mathcal{C}^1_{a,b}$, $\eta_a \circ G(f) = F(f) \circ \eta_b$*. A* natural isomorphism *is a natural transformation* η *for which each* η_a *is an isomorphism.*

If η is a natural isomorphism from F to G, the maps η_a^{-1} give a natural isomorphism η^{-1} from G to F.

The previous equality is often expressed in the form of a *commutative diagram*.

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ \downarrow{\scriptstyle \eta_a} & & \downarrow{\scriptstyle \eta_b} \\ G(a) & \xrightarrow{G(f)} & G(b) \end{array}$$

The convention for commutative diagrams is, any two compositions of morphisms with the same source and target are equal.

Example 5.17. *In Example 5.15,* F *is, by construction, naturally isomorphic to the identity functor. In particular the maps* η_a *define a natural isomorphism from* Id *to* F*.*

Definition 5.18. *Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exist functors $F : C \to D$ and $G : D \to C$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to identity functors.*

Below, we'll often prove this directly; constructing the functors helps understand how to translate from one formalism to another.

Because this definition is a bit notionally cumbersome, we'll use an alternate, equivalent definition of equivalence that is intuitive and eleminates the need to understand the details of natural transformations.

Definition 5.19. *A* full functor *is a functor F such that each $F^1_{a,b}$ is surjective.*

A faithful functor *is a functor F such that each $F^1_{a,b}$ is injective.*

An essentially injective functor *is a functor $F : \mathcal{C} \to \mathcal{D}$ such that for each $a, b \in \mathcal{C}^0$, if $F^0(a) \cong F^0(b)$, then $a \cong b$.*

An essentially surjective functor *is a functor $F : \mathcal{C} \to \mathcal{D}$ such that for each $d \in \mathcal{D}^0$, there is an object $c \in \mathcal{C}^0$ such that $F^0(c) \cong d$.*

A functor is essentially bijective *if it is both essentially injective and essentially surjective.*

Proposition 5.20. *Categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there exists a full, faithful, essentially bijective functor $F : C \to D$.*

Example 5.21. *Given any category $\mathcal{C}$, a* skeleton *$\mathcal{C}_{\mathcal{SKEL}}$ of that category can be constructed as follows: choose one object from each isomorphism class $\mathcal{C}^0$, and construct the full subcategory $\mathcal{C}_{\mathcal{SKEL}}$ containing the choices. The inclusion functor is then full, faithful, and essentially bijective, so $\mathcal{C}$ is equivalent as a category to $\mathcal{C}_{\mathcal{SKEL}}$.*

6 Equivalent hypergraph categories

6.1 Minor variants

Proposition 6.1. *Categories $\mathcal{HG}$ and $\mathcal{OHG}$ are equivalent.*

Proof. For each set X, choose a canonical total ordering, producing a totally ordered set X_e. Given a permutation π on elements of X, let X_π denote the totally ordered set that results from applying π to X_e. Note that every total ordering of X is X_π for some π.

Let $F : \mathcal{OHG} \to \mathcal{HG}$ be the forgetful functor, i.e. $F^0(V_\pi, E_\phi, I_{\phi,\pi}) = (V, E, I)$, where I is the incidence function induced from $I_{\phi,\pi}$. The morphisms on $\mathcal{OHG}$ are just hypergraph morphisms on the underlying sets for vertices and edges; let F^1 send morphisms to themselves.

Since each $F^1_{a,b}$ is a bijection, F is a full, faithful functor. If $F^0(a) \cong F^0(b)$, then there is a permutation of objects (and underlying permutation map) that takes a to b. Thus F is essentially injective. F is surjective on objects and thus essentially surjective. By Proposition 5.20, $\mathcal{OHG}$ and $\mathcal{HG}$ are equivalent as categories. $\square$

Discussion: This equivalence is really the simplest possible; in $\mathcal{OHG}$ we have decorated the objects in $\mathcal{HG}$ with an extra property, an ordering function, which we don't use in the definition of the morphisms.

Definition 6.2. *Let $\mathcal{HG}_{\mathcal{BMAT}}$ be the full subcategory of $\mathcal{HG}$ consisting of hypergraphs constructed from boolean matrices, as in Example 4.4.*

Proposition 6.3. *$\mathcal{HG}_{\mathcal{BMAT}} \cong \mathcal{HG}$ as categories.*

Proof. Since $\mathcal{HG}_{\mathcal{BMAT}}$ is a full subcategory, the inclusion functor is full, faithful, and essentially injective. Given any hypergraph, it is easy to construct an isomorphic hypergraph in $\mathcal{HG}_{\mathcal{BMAT}}$, by ordering its vertices and edges in some way and computing the corresponding incidence matrix. Thus the inclusion functor is essentially surjective. By Proposition 5.20, $\mathcal{HG}_{\mathcal{BMAT}} \cong \mathcal{HG}$. □

A binary relation is a triple (X, Y, R) where $R \subset X \times Y$. For each relation R on X, Y there is a characteristic function $\chi(R) : X \times Y \to \{0, 1\}$, and for each incidence function I on $X \times Y$ a characteristic subset $\chi^{-1}(I) = I^{-1}(1)$. Binary relations on $V \times E$ and hypergraphs really only differ in that instead whereas the relation defines a subset of $X \times Y$; the hypergraph incidence function defines the characteristic function of a relation.

Definition 6.4. *Define a morphism of binary relations $f : (X_1, Y_1, R_1) \to (X_2, Y_2, R_2)$ as a pair of functions (f_X, f_Y), $f_X : X_1 \to X_2$, $f_Y : Y_1 \to Y_2$ such that (f_X, f_Y) is a morphism of hypergraphs $(X_1, Y_1, \chi(R_1)) \to (X_2, Y_2, \chi(R_2))$.*

Let $\mathcal{HG}_{\mathcal{BINREL}}$ be the category of binary relations and morphisms of binary relations.

Proposition 6.5. *The categories $\mathcal{HG}$ and $\mathcal{HG}_{\mathcal{BINREL}}$ are equivalent.*

Proof. Define $F : \mathcal{HG} \to \mathcal{HG}_{\mathcal{BINREL}}$ such that $F^0((X, Y, I)) = (X, Y, \chi(I))$ and morphisms are fixed.

Then F is a functor and has a strict inverse: let G similarly send (X, Y, R) to $(X, Y, \chi^{-1}(R))$ and fix morphisms. Then both double compositions are identity functors on the nose.

Thus $\mathcal{HG}$ and $\mathcal{HG}_{BINREL}$ are equivalent. □

In Definition 6.4, we could define morphisms directly in terms of relations R_1 and R_2. Unless we have an independent notion of what morphisms should be for binary relations, the way this translation is performed is to choose morphisms in such a way that F^1 can be constructed so as to give an equivalence.

The result of this re-expression is a pair of maps $f_X : X \to X'$ and $f_Y : Y \to Y'$ that form a morphism of relations (X, Y, R) and (X', Y', R') if for all $y \in Y$, $f_X(\{x \in X | (x, y) \in R\}) = \{x \in X' | (x', f(y)) \in R'\}$.

For binary relations, this formulation is odd, as it is not symmetric in X and Y. In particular the dual given by switching Y and X is not a functor.

A more natural-seeming choice for a morphism of relations is this: a morphism of relations (X, Y, R) and (X', Y', R') is a pair of functions $f_X : X \to X'$ and $f_Y : Y \to Y'$ such that for all $(x, y) \in X \times Y$, $(x, y) \in R \Rightarrow (f_X(x), f_Y(y)) \in R'$. If we translate this requirement back into hypergraph language this condition is just Example 5.6.

Definition 6.6. *Let $\mathcal{HG}_\sqcup$ be the complete subcategory of $\mathcal{HG}$ consisting of hypergraphs (V, E, I) in which V and E are disjoint.*

Proposition 6.7. *$\mathcal{HG}_\sqcup$ and $\mathcal{HG}$ are equivalent categories.*

Proof. Define $F : \mathcal{HG} \to \mathcal{HG}_\sqcup$ as follows: Given $(V, E, I) \in \mathcal{HG}^0$, let

$$F^0((V, E, I)) = (V \otimes \{0\}, E \otimes \{1\}, \bar{I}),$$

where $\bar{I} : (V \times \{0\}, E \otimes \{1\}) \to \{0, 1\}$ such that $\bar{I}((v, 0), (e, 1)) = I(v, e)$.
Given $f : (V_1, E_1, I_1) \to (V_2, E_2, I_2)$, $f = (f_V, f_E)$, define

$$F^1_{(V_1,E_1,I_1),(V_2,E_2,I_2)}(f) = (f_V \times \mathrm{id}_{\{0\}}, f_E \times \mathrm{id}_{\{1\}}),$$

It is easily checked that F is a functor, and the following hold:

1. If $V \sqcup E = \emptyset$, $F^0((V, E, I)) \cong (V, E, I))$. Thus F is essentially surjective.
2. An isomorphism $g : F^0((V, E, I)) \to F^0((V', E', I'))$ is the image of an isomorphism $f : (V, E, I) \to (V', E', I')$. Thus F essentially injective.
3. F^1 maps morphism sets bijectively, and is thus a full, faithful functor.

By Proposition 5.20, $\mathcal{HG}$ and $\mathcal{HG}_\sqcup$ are equivalent categories. □

6.2 Lax hypergraph categories

All of the above proofs work if the replacement of $\mathcal{HG}$ wih $\mathcal{HG}'$ from Definition 5.9 is propagated to the other definitions. One obtains categories $\mathcal{OHG}'$, $\mathcal{HG}'_{\mathcal{BMAT}}$, etc. which are equivalent to $\mathcal{HG}'$.

Proposition 6.8. *$\mathcal{HG}$ and $\mathcal{HG}'$ are not equivalent as categories.*

Proof. [4] A bit of consideration makes it clear that there's no obvious way to construct a functor which is both full and faithful. However, the flexibility of our notion of equivalance works against us here; there's no guarantee that an equivalence would map objects in a one-to-one or onto manner so it's hard to show we can't form an object-correspondence in any way that the morphisms respect. The typical way to prove inequivalence of categories is to find properties of morphisms which are invariant under equivalence and show that they are different in the two categories. The following definitions are all preserved when one passes through an equivalence of categories; we leave the proofs as exercises.

Definition 6.9. *In a category $\mathcal{C}$, a morphism $f : b \to c$ is* monic *if for any object a and any two morphisms $g, h : a \to b$, whenever $g \circ f = h \circ f$, $h = f$.*

[4] In response to an incorrect proof in an early draft, the following approach was suggested by William Grilliette.

Let $H_v = \{\{v\}, \{\}\}$, a hypergraph consisting of a single vertex v and no edges. Such a hypergraph exists in both $\mathcal{HG}$ and $\mathcal{HG}'$. Given a monic hypergraph morphism $f : H_1 \to H_2$, in either category, if H_1 has multiple vertices we can construct maps $g, h : H_v \to H_1$ which map v to any desired vertex in H_1. If g, h are different, $g \circ f$ and $h \circ f$ must be different. Thus a monic morphism must map vertices injectively in both $\mathcal{HG}$ and $\mathcal{HG}'$. A similar argument shows that monic maps map hyperedges injectively in both categories, and that these two conditions are sufficient to guarantee that f is monic in either case.

The maps themselves, however, are different between the two categories, because we don't need to map the vertices of each edge surjectively in $\mathcal{HG}'$.

Definition 6.10. *In a category $\mathcal{C}$, a morphism $f : b \to c$ is* epic *if for any object a and any two morphisms $g, h : a \to b$, whenever $f \circ g = f \circ h$, $h = f$.*

Similar consideration shows that in either category, a morphism is epic if it maps both vertices and hyperedges surjectively.

Definition 6.11. *A morphism which is both monic and epic is a* bimorphism*.*

In HG, every bimorphism $f : H_1 \to H_2$ is an isomorphism, which can be seen as follows. Suppose f is a bimorphism. Then f is both monic and epic. Thus, every hyperedge of H_1 is mapped by f to a hyperedge of the same cardinality in H_2 with an associated bijection on the vertices. We can invert the map by inverting the vertex map along with the associated edge map.

In HG', however, a bimorphism $f : H_1 \to H_2$ is not guaranteed to map the vertices in edge e_1 to the complete set of vertices in edge $f_e(e_1)$. For example, the hypergraph consisting of a single isolated vertex and single 0-hyperedge maps via a bimorphism to the graph consisting of a single vertex and single 1-hyperedge. This map is not an isomorphism because we cannot construct an inverse map; we can't map an edge containing an element to an empty edge. Thus $\mathcal{HG}' \not\cong \mathcal{HG}$.

□

6.3 Bicolored graphs

Definition 6.12. *A* bicolored graph *is a tuple $G = (V, E, c)$, where (V, E) is an undirected graph and $c : V \to \{0, 1\}$ is a coloring of vertices such that for any $(v_1, v_2) \in E$, $c(v_1) \neq c(v_2)$. A* (surjective) bicolored graph homomorphism *$f : (V_1, E_1, c_1) \to (V_2, E_2, c_2)$ is a (surjective) graph homomorphism such that for each $i \in \{0, 1\}$, $f(c_1^{-1}(i)) \subset c_2^{-1}(i)$. In other words, a homomorphism that preserves colors.*

Definition 6.13. *In [5], the category $\mathcal{BIC}_\mathcal{S}$ (which is there called $\mathcal{BIC}$) is defined as the category of bicolored graphs, with surjective bicolored graph homomorphisms.*

We use the notation $\mathcal{BIC}$ to define the category of bicolored graphs, with (not necessarily surjective) bicolored graph homomorphisms.

The natural correspondence between bicolored graphs and hypergraphs is to map hypergraph vertices to one color of bicolored graph vertex, edges to the other color, and incidences to edges.

Under this mapping, however, our notion of hypergraph homomorphism does not line up with the morphisms in either of these categories. In the non-surjective bicolored homomorphism, there is no guarantee that the vertices of an edge will be mapped surjectively to the vertices of an edge. In the surjective bicolored homomorphism, the map is required to be surjective on vertices.

The point of requiring surjectivity on vertices is to ensure that the dual hypergraph construction is a functor. For this purpose, a slightly weaker condition would be sufficient.

Definition 6.14. *The* neighborhood $N(G, v)$ *of a graph* $G = (V, E)$ *at* $v \in V$ *is the induced subgraph of* G *containing all vertices* v' *contained in an edge* e *which also contains* v*.*

Let $f : G_1 \to G_2$ *be a graph homomorphism, with* $G_1 = (V_1, E_1)$*. We say that* f *is locally injective (surjective) at a vertex* $v \in V_1$ *if the induced map* $f_v : N(G_1, v) \to N(G_2, f(v))$ *is injective (surjective).*

Definition 6.15. *Given a hypergraph* $H = (V, E, I)$*, define its associated bicolored graph* $G_H = (V \bigsqcup E, \overline{I}, c)$*, with* $\overline{I} \subset V \bigsqcup E \times V \bigsqcup E$ *such that* $(x, y) \in \overline{I}$ *iff*

- $x \in V$*,* $y \in E$ *and* $I(x, y) = 1$*, OR*
- $x \in E$*,* $y \in V$ *and* $I(x, y) = 1$*,*

and $c : V \bigsqcup E \to \{0, 1\}$ *such that* $c^{-1}(0) = V$*,* $c^{-1}(1) = E$*.*

Conversely, given a bicolored graph $G = (V, E, c)$*, define the hypergraph*

$H_G = (c^{-1}(0), c^{-1}(1), \chi$*, where* $\chi : c^{-1}(0) \times c^{-1}(1) \to \{0, 1\}$ *such that* $\chi(v, e) = 1$ *iff* $(v, e) \in E$*.*

Proposition 6.16. *The above constructions give a bijective correspondence between bicolored graphs and hypergraphs.*

Definition 6.17. *Given a hypergraph morphism* $f : H_1 \to H_2, f = (f_V, f_E), f_V : V_1 \to V_2, f_E : E_1 \to E_2$*, define* $\phi_{H_1,H_2}(f) : V_1 \bigsqcup E_1 \to V_2 \bigsqcup E_2$ *such that* $f = (f_V \bigsqcup f_E)$*.*

Proposition 6.18. $\phi_{H_1,H_2}(f)$ *is a bicolored graph homomorphism* $G_{H_1} \to G_{H_2}$*.*

Proof. For simplicity of notation when referring to disjoint copies, assume V_i and E_i are already disjoint.

Preservation of colors of vertices is obvious. For $v \in V_1 = c^{-1}(0)$, $e \in E_1 = c^{-1}(1)$, suppose (v, e) is an edge in G_{H_1}. This holds if and only if v and e are incident in H_1. Then $f_V(v)$ and $f_E(e)$ are incident in H_2, because hypergraph homomorphisms preserve incidences. Thus $(f(v), f(e))$ is an edge in G_{H_2}. Thus $\phi_{H_1,H_2}(f)$ is a graph homomorphism. Thus $\phi_{H_1,H_2}(f)$ is a bicolored graph homomorphism. □

Proposition 6.19. *Let* $G_1 = (V_1, E_1, c_1)$ *and* $G_2 = (V_2, E_2, c_2)$ *be bicolored graphs, and let* $f : G_1 \to G_2$ *be a bicolored graph homomorphism. Then* $f = \phi_{H_{G_1},H_{G_2}}(\bar{f})$ *for some hypergraph homomorphism* $\bar{f} : H_{G_1} \to H_{G_2}$ *if and only if* f *is locally surjective on* $c_1^{-1}(1)$*.*

Proof. Suppose f is the image $\bar{f} : H_{G_1} \to H_{G_2}$ under ϕ. Then f must be locally surjective for all $e \in c_1^{-1}1$, as this is the image the condition that vertices of hyperedges must be mapped surjectively to vertices of mapped hyperedges by $\bar{f}$. Conversely, suppose f is locally surjective on $c_1^{-1}(1)$. Let $V_1 = c_1^{-1}(0), E_1 = c_1^{-1}(1), V_2 = c_2^{-1}(0), E_2 = c_2^{-1}(1)$. Define $f_V : V_1 \to V_2$ such that $f_V = f|_{V_1}$, $f_E : E_1 \to E_2$ such that $f_E = f|_{E_1}$. Then $f = \phi_{H_{G_1},H_{G_2}}((f_V, f_E))$. Using the same notations as in Definition 5.4, we have, for $E \in E_1$

$$f_V(e) = f_V(\{v \in V_1 | v \in e\}) = f_V(\{v \in V_1 | (v, e) \in I_1\} = \{v \in V_2 | (v, f(e) \in I_2\}$$

since f is locally surjective. But

$$\{v \in V_2 | (v, f(e) \in I_2\} = V(f(e)).$$

Thus $\bar{f} = (f_V, f_E)$ is a hypergraph morphism. □

Definition 6.20. *Let $\mathcal{BIC}_{\mathcal{LS}1}$ be the category of bicolored graphs, with morphisms $f : (V_1, E_1, c_1) \to (V_2, E_2, c_2)$ which are locally surjective on $c_1^{-1}(1)$.*

Proposition 6.21. *$\mathcal{HG}$ and $\mathcal{BIC}_{\mathcal{LS}1}$ are equivalent as categories.*

Proof. Define $F = (F_0, F_1) : \mathcal{HG} \to \mathcal{BIC}_{\mathcal{LS}1}$ such that $F_0(H) = G_H$, and given $f : H_1 \to H_2$, $F_1(f) = \phi_{H_1,H_2}(f)$.
Then

$$F_1(Id_H) = F_1((Id_V, Id_E)) = Id_{V \sqcup E} = Id_{G_H}.$$

Also,

$$F_1(f \circ g) = F_1((f_V \circ g_V, f_E \circ g_E)) = (f_V \circ g_V) \sqcup (v_E \circ g_E) = (f_V \sqcup f_E) \circ (g_V \sqcup g_E) = F_1(f) \circ F_1(g).$$

Thus F is a functor. Since F_0 is a bijection on objects and F_1 is bijective on morphism spaces, F is fully faithful and essentially surjective. Thus F is an isomorphism of categories. □

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