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PNNL-35764

Bijection Theorems for Property Hypergraphs

March 2024

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U.S. DEPARTMENT OF
ENERGY

Prepared for the U.S. Department of Energy
under Contract DE-AC05-76RL01830

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UNITED STATES DEPARTMENT OF ENERGY
under Contract DE-AC05-76RL01830

Printed in the United States of America

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Acknowledgments

Special thanks to the KBASE team, in particular Emilie Purvine, Brett Jefferson, Sinan Aksoy, and Bill Kay for helpful discussions and feedback.

Definition 1. *A hypergraph H is a triple $H = (V, E, i)$, where V and E are sets, known as the* vertex set *and the* edge set *respectively, and $i : V \times E \to \{0, 1\}$, known as the* incidence function*.*

The set $i^{-1}(1) \subset V \times E$ is known as the incidence relation *for H.*

Definition 2. *A property hypergraph is tuple $(V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$, where (V, E, i) is a hypergraph, and $\phi : V \to T_V$, $\epsilon : E \to T_E$, and $\iota : i^{-1}(1) \to T_i$, for some finite sets T_V, T_E, T_i, called the set of* vertex types*,* edge types*, and* incidence types *respectively.*

Even when V and E are not disjoint, the incidence relation distinguishes vertices from edges within the context of an incidence.

Definition 3. *Let $H = (V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$ be a property hypergraph. The dual property hypergraph is a tuple $H^* = (E, V, i^*, \epsilon, \phi, \iota^*, T_E, T_V, T_i)$, where $i^* : E \times V \to \{0, 1\}$ such that $i^*(e, v) = i(v, e)$, and $\iota^* : (i^*)^{-1}(1) \to T_i$ such that $\iota^*(e, v) = \iota(v, e)$.*

Note that ι^* and ι are different. This creates a somewhat unsatisfying situation when we try to claim an equivalence between hypergraphs and bicolored graphs in the case when each has properties. In an undirected bipartite graph, an edge property does not specify which of the edge's vertices plays which role in the property. We could declare that the semantics is determined by the property map and the bicoloring in tandem, which would work mathematically, but in real data, which often follows subject-object property semantics but often does not admit a bicoloring, it seems more natural to associate an incidence property with a directed edge. The direction of an edge in our directed property graphs indicates the roles of vertices in the incidence property, and can be reversed by altering the semantics of the property definition.

Definition 4. *A* directed bicolored graph *is a tuple $G = (V, E, c)$, where V is a set, called the set of* vertices*, $E \subset V \times V$ is called the set of* directed edges*, and $c : V \to \{0, 1\}$ such that for any $(v_1, v_2) \in E$, $c(v_1) \neq c(v_2)$.*

Definition 5. *A directed bicolored property graph is a tuple $G = (V, E, c, \phi, \epsilon, T_V, T_E)$, where (V, E, c) is a directed bicolored graph and $\phi : V \to T_V$, $\epsilon : E \to T_E$, for some finite sets T_V and T_E, called the* vertex property set *and the* edge property set *respectively.*

An undirected property graph with bicoloring-determined semantics may be replaced with a directed graph by directing each edge from the first color to the second and redefining the property semantics in terms of the edge direction.

From now on, we will assume that all graphs are directed.

Definition 6. *Let $H = (V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$ be a property hypergraph, and suppose $V \cap E = \emptyset$. Then hypergraph (V, E, i) is equivalent to the bicolored graph $G = (V', E', c')$, where*

1. *$V' = V \cup E$,*
2. *$E' = i^{-1}(1)$, and*
3. *$c' : V' \to \{0, 1\}$ such that $(c')^{-1}(0) = V$ and $(c')^{-1}(1) = E$.*

Define $G_H = (V', E', c', \phi', \epsilon', T_{V'}, T_{E'})$, where

1. *$T_{V'} = T_V \cup T_E$,*
2. *$T_{E'} = T_i$,*
3. *$\phi' : V' \to T_{V'}$ such that $\phi'|_V = \phi$ and $\phi'|_E = \epsilon$, and*
4. *$\epsilon' : E' \to T_{E'}$ such that $\epsilon' = \iota$.*

Then G_H is a (directed) bicolored property graph, with all edges pointing from $(c')^{-1}(0)$ to $(c')^{-1}(1)$.

Definition 7. *Conversely, suppose $G = (V, E, c, \phi, \epsilon, T_V, T_E)$ is a (directed) bicolored property graph. Define $H_G = (V', E', i', \phi', \epsilon', \iota', T'_V, T'_E, T'_i)$, where:*

1. $V' = (c)^{-1}(0)$,
2. $E' = (c)^{-1}(1)$,
3. $i' : V' \times E' \to \{0, 1\}$ *is the restriction to* $V' \times E'$ *of the adjacency function for* G.
4. $T'_V = \phi(V')$,
5. $T'_E = \phi(E')$,
6. $T'_i = T_E$,
7. $\phi' : V' \to T'_V$ *such that* $\phi' = \phi|_{V'}$,
8. $\epsilon' : E' \to T'_E$ *such that* $\epsilon' = \phi|_{E'}$,
9. $\iota' : i'^{-1}(1) \to T'_i$ *such that* $\iota'((v', e')) = \epsilon((v', e'))$.

Then H_G is a property hypergraph.

Proposition 8. *Let $H = (V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$ be a property hypergraph such that the functions ϕ and ϵ are surjective. Then $H_{G_H} = H$.*

Proof. We must have $G_H = (V', E', c', \phi', \epsilon', T_V \cup T_E, T_i)$ for some V', E', c', ϕ', and ϵ', and also $H_{G_H} = (V'', E'', i'', \phi'', \epsilon'', \iota'', T_{V''}, T_{E''}, T_i)$ for some V'',E'', i'', ϕ'', ϵ'', and ι''. Then by the above definitions:

1. $V'' = (c')^{-1}(0) = V$,
2. $E'' = (c')^{-1}(1) = E$,
3. $\mathrm{dom}(i'') = V'' \times E'' = V \times E$,
4. for $(v, e) \in V'' \times E''$, $i''(v, e) = i(v, e)$,
5. $\phi'' = \phi'|_{V''} = \phi$,
6. $T_{V''} = \phi'(V'') = \phi'((c')^{-1}(0)) = \phi'(V) = \phi(V) = \mathrm{range}(\phi) = T_V$, since ϕ is surjective,
7. $\epsilon'' = \phi'|_{E''} = \epsilon$,
8. $T_{E''} = \phi'(E'') = \phi'((c')^{-1}(1)) = \phi'(E) = \epsilon(E) = \mathrm{range}(\epsilon) = T_E$, since ϵ is surjective,
9. $\iota'' = \epsilon' = \iota$, and
10. $T''_i = T'_E = T_i$.

□

Note that if ϕ and ϵ are not surjective, the bicolored property graph formulation offers no natural way to store their individual codomains in G_H, as only the combined codomain of ϕ' is preserved.

If directed property graph arrows serve only to indicate how to read the semantics of the property, for formal purposes it makes sense to direct the arrows with a consistent convention and interpret the semantics accordingly. If it is possible to do this in a way that respects a bicoloring, there is an equivalent hypergraph:

Proposition 9. *Let $G = (V, E, c, \phi, \epsilon, T_V, T_E)$ be a (directed) bicolored property graph, with all edges pointing from $c^{-1}(0)$ to $c^{-1}(1)$, and ϕ surjective. Then $G_{H_G} = G$.*

Proof. We have $H_G = (V', E', i', \phi', \epsilon', \iota', T'_V, T'_E, T'_i)$ and $G_{H_G} = (V'', E'', c'', \phi'', \epsilon'', T''_V, T''_E)$ for some assignments of these variables. The proof is similar to above:

1. $V'' = V' \cup E' = c^{-1}(0) \cup c^{-1}(1) = V$,
2. $E'' = (i')^{-1}(1) = \{(v, e) \in V' \times E' | (v, e) \in E\} = E$,
3. $c''(v) = 0$ if $v \in V'$ and $c''(v) = 1$ if $v \in E'$ then $c'' = c$,
4. $\phi''|_{V'} = \phi' = \phi|_{C^{-1}(0)}$ and $\phi''|_{E'} = \epsilon' = \phi|_{C^{-1}(1)}$ then $\phi'' = \phi$,
5. $\epsilon'' = \iota' = \epsilon$,
6. $T''_V = T'_V \cup T'_E = \phi(V') \cup \phi(E') = \phi(V) = T_V$, since ϕ is surjective,
7. $T'_i = T_E$, and
8. $T''_E = T'_i$.

□

For a property hypergraph H with V and E not disjoint, there is generally no way to map it to a property graph on vertex set $V \cup E$, bicolored or not. This is because for $t \in V \cap E$, we may have $\phi(t) \neq \epsilon(t)$, so there is no way to consistently define $\phi'(t)$ in G_H. If, however, there is a well-defined property map on $V \cup E$, and a well-defined incidence map on $(V \cup E) \times (V \cup E)$, the construction in Definition 6 is well-defined, except that we cannot define a bicoloring. In the opposite direction, given a property graph without coloring, we can construct a property hypergraph, with vertices and edges not necessarily disjoint:

Definition 10. *Given a (directed) property graph $G = (V, E, \phi, \epsilon, T_V, T_E)$, we can define a property hypergraph $H_G = (V', E', i', \phi', \epsilon', \iota', T'_V, T'_E, T'_i)$, where*

1. $V' = \{v \in V | \exists_{v_1 \in v} (v, v_1) \in E\}$,
2. $E' = \{v \in V | \exists_{v_1 \in v} (v_1, v) \in E\}$,
3. *i' is the adjacency function for G restricted to $V' \times E'$,*
4. $T'_V = \phi(V')$,
5. $T'_E = \phi(E')$.
6. $\phi' : V' \to T'_V, \phi' = \phi|_{V'}$,
7. $\epsilon' : E' \to T'_E, \epsilon' = \phi|_{E'}$,
8. *$\iota' = \epsilon$, and*
9. $T'_i = T_E$.

Then H_G is a property hypergraph.

When G happens to admit a bicoloring compatible with the directed edges, this construction reduces to the ordinary bipartite-graph to hypergraph correspondence. Though we do not get into the details here, the map $G \to H_G$ extends to a morphism from the category of graphs to the category of hypergraphs, under which the canonical map $G' \to G$ which maps the two-fold directed cover G' of G onto G has as its functorial image the canonical map $H_{G'} \to H_G$ which maps the disjoint union hypergraph $H'_G = H_{G'}$ onto H_G.

Proposition 11. *Let $G = (V, E, \phi, \epsilon, T_V, T_E)$ be a (directed) property graph with no degree-zero vertices, and ϕ surjective. Let $H_G = (V', E', i', \phi', \epsilon', \iota', T'_V, T'_E, T'_i)$ defined as above. Then $G_{H_G} = G$.*

Proof. Let $G_{H_G} = (V'', E'', \phi'', \epsilon'', T''_V, T''_E)$ Then

1. $V'' = V' \cup E' = V$, since G has no degree-zero vertices,
2. $E'' = (i')^{-1}(1) = E$,
3. $\phi''|_{V'} = \phi'$ and $\phi''|_{E'} = \epsilon'$ and $\phi' = \phi|_{V'}$ and $\epsilon' = \phi|_{E'}$ and so $\phi'' = \phi$ which is well-defined since the functions ϕ' and ϵ' agree on their intersection,
4. $\epsilon'' = \iota' = \epsilon$,
5. $T''_V = T'_V \cup T'_E = \phi(V') \cup \phi(E') = \phi(V) = T_V$, since ϕ is surjective.

□

In the event that the direction of a graph's oriented arrows does not match the bipartite hierarchy determined by the edge semantics, the two may be treated separately. The starting definitions would be as follows:

Definition 12. *A property hypergraph with orientation is tuple $(V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$, where (V, E, i) is a hypergraph, and $\phi : V \to T_V$, $\epsilon : E \to T_E$, and $\iota : i^{-1}(1) \to T_i \times \mathbb{Z}_2$, for some sets T_V, T_E, T_i, called the set of* vertex types*,* edge types*, and* incidence types *respectively.*

Definition 13. *Let $H = (V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$ be a* property hypergraph with orientation*, and suppose $V \cap E = \emptyset$. Then hypergraph (V, E, i) is equivalent to the bicolored graph $G = (V', E', c')$, where*

1. *$V' = V \cup E$,*
2. *$E' = \{(v, e) : (v, e) \in i^{-1}(1), \iota(v, e) = 0\} \cup \{(e, v) : (v, e) \in i^{-1}(1), \iota(v, e) = 1\}$, and*
3. *$c' : V' \to \{0, 1\}$ such that $(c')^{-1}(0) = V$ and $(c')^{-1}(1) = E$.*

Define $G_H = (V', E', c', \phi', \epsilon', T_{V'}, T_{E'})$, where

1. *$T_{V'} = T_V \cup T_E$,*
2. *$T_{E'} = T_i$,*
3. *$\phi' : V' \to T_{V'}$ such that $\phi'|_V = \phi$ and $\phi'|_E = \epsilon$, and*
4. *$\epsilon' : E' \to T_{E'}$ such that $\epsilon' = \iota$.*

Then G_H is a (directed) bicolored property graph*.*

Definition 14. *Conversely, suppose $G = (V, E, c, \phi, \epsilon, T_V, T_E)$ is a (directed) bicolored property graph. Define $H_G = (V', E', i', \phi', \epsilon', \iota', T'_V, T'_E, T'_i)$, where:*

1. *$V' = (c)^{-1}(0)$,*
2. *$E' = (c)^{-1}(1)$,*
3. *$i' : V' \times E' \to \{0, 1\}$ is the restriction to $V' \times E'$ of the adjacency function for G.*
4. *$T_{V'} = \phi(V')$,*
5. *$T_{E'} = \phi(E')$,*
6. *$T_{i'} = T_E$*
7. *$\phi' : V' \to T_{V'}$ such that $\phi' = \phi|_{V'}$,*
8. *$\epsilon' : E' \to T_{E'}$ such that $\epsilon' = \phi|_{E'}$, and*

9. $\iota' : i'^{-1}(1) \to T_{i'} \times \mathbb{Z}_2$ *such that* $\iota'(v', e') = (\epsilon(v', e'), \mathbb{1}_{(v',e') \in E})$.

Then H_G is a property hypergraph with orientation.

Proposition 15. *Let $H = (V, E, i, \phi, \epsilon, \iota, T_V, T_E, T_i)$ be a property hypergraph with orientation such that the functions ϕ and ϵ are surjective. Then $H_{G_H} = H$.*

Proof. We must have $G_H = (V', E', c', \phi', \epsilon', T_V \cup T_E, T_i)$ for some V', E', c', ϕ', and ϵ', and also $H_{G_H} = (V'', E'', i'', \phi'', \epsilon'', \iota'', T_{V''}, T_{E''}, T_i)$ for some V'',E'', i'', ϕ'', ϵ'', and ι''. Then by the above definitions:

1. $V'' = (c')^{-1}(0) = V$,
2. $E'' = (c')^{-1}(1) = E$,
3. $\mathrm{dom}(i'') = V'' \times E'' = (c')^{-1}(0) \times (c')^{-1}(1) = V \times E$,
4. for $(v, e) \in V'' \times E''$, $i''(v, e) = i(v, e)$,
5. $\phi'' = \phi'|_{V''} = \phi$,
6. $T_{V''} = \phi'(V'') = \phi'((c')^{-1}(0)) = \phi'(V) = \phi(V) = \mathrm{range}(\phi) = T_V$, since ϕ is surjective,
7. $\epsilon'' = \phi'|_{E''} = \epsilon$,
8. $T_{E''} = \phi'(E'') = \phi'((c')^{-1}(1)) = \phi'(E) = \epsilon(E) = \mathrm{range}(\epsilon) = T_E$, since ϵ is surjective,
9. for $(v'', e'') \in E'$, $\iota''(v'', e'') = \epsilon'(v'', e'') = \iota(v'', e'')$ and for $(e'', v'') \in E'$, $\iota''(v'', e'') = \epsilon'(e'', v'') = \iota(v'', e'')$, and
10. $T_i'' = T_{E'} = T_i$.

□

Proposition 16. *Let $G = (V, E, c, \phi, \epsilon, T_V, T_E)$ be a (directed) bicolored property graph such that the function ϕ is surjective. Then $G_{H_G} = G$.*

Proof. We have $H_G = (V', E', i', \phi', \epsilon', \iota', T_V', T_E', T_i')$ and $G_{H_G} = (V'', E'', c'', \phi'', \epsilon'', T_V'', T_E'')$ for some assignments of these variables. The proof is similar to above:

1. $V'' = V' \cup E' = c^{-1}(0) \cup c^{-1}(1) = V$,
2. $E'' = \{(v, e) \in V' \times E' | (v, e) \in E\} \cup \{(e, v) \in V' \times E' | (e, v) \in E\} = E$,
3. $c''(v) = 0$ if $v \in V'$ and $c''(v) = 1$ if $v \in E'$ then $c'' = c$,
4. $\phi''|_{V'} = \phi' = \phi|_{c^{-1}(0)}$ and $\phi''|_{E'} = \epsilon' = \phi|_{c^{-1}(1)}$ then $\phi'' = \phi$,
5. $\epsilon'' = \iota' = \epsilon$,
6. $T_V'' = T_V' \cup T_E' = \phi(V') \cup \phi(E') = \phi(V) = T_V$, since ϕ is surjective,
7. $T_i' = T_E$, and
8. $T_E'' = T_i'$.

□

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