Multi-Interval Elicitation of Random Intervals for Engineering Reliability Analysis

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Abstract

Random intervals are increasingly useful in engineering modeling, but are difficult to measure and elicit from experts. We present a method for constructing random intervals by eliciting simple "multi-interval" and trace information from investigators. By eliciting in addition to "how high" and "how low", simply also "how high and low might the min and max themselves be", we generate an equivalence class of possibility distributions, and in turn a canonical member of an equivalence class of random intervals.

1. Introduction

A random interval is a interval-valued random variable, or a Dempster-Shafer evidence structure on the Borel field. Random interval approaches are an emerging technology for engineering reliability analysis [2, 6, 7, 8]. Their great advantage is their ability to represent not only randomness via probability theory, but also imprecision and nonspecificity via intervals, in a overall mathematical structure which is close to optimally simple. As such, they are superb ways for engineering modelers to approach the world of Generalized Information Theory (GIT), whose goal is to complement probability theory with a range of other techniques such as possibility measures, fuzzy sets and logic, belief and plausibility measure, etc.

Random intervals can be difficult, be combinatorially complex, and presenting challenges to modelers and investigators in their elicitation and interpretation. In particular, interpreting the fundamental structures of a random interval (focal elements, basic probability weights, cumulative plausibility and belief as bounds on a CDF, etc.) can be a daunting task for the content expert. It is desirable to interact with investigators over more familiar mathematical objects.

In this paper we present a method for constructing complex random interval representations by eliciting simple "multi-interval" information from investigators. Central to our method is the fact that random intervals generate simpler structures, including possibility distributions and fuzzy sets. Most simple is the single interval, easily interpreted as "how high and low can a certain quantity be?". By eliciting in addition to "how high" and "how low", simply also "how high and low might the min and max themselves be", we derive a multi-interval, which generates an equivalence class of possibility distributions (effectively, a rough set on $\mathbb{R}$), and in turn a canonical member of an equivalence class of random intervals.

2 Mathematical Preliminaries

2.1 Random Set Notation

Throughout the paper assume a universe of discourse $\Omega = \{\omega\}$, with cardinality to be specified. Given a finite class $\mathcal{C} = \{A\} \subseteq 2^\Omega$, define the **support** as $\mathbf{U}(\mathcal{C}) := \bigcup_{A \in \mathcal{C}} A$.

A function $m \colon 2^\Omega \mapsto [0,1]$ is an **evidence function** (basic probability assignment) when $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$. Given an evidence function m, then

$$\mathcal{S} := \{\langle A_j, m_j \rangle : m_j > 0, 1 \le j \le N\},$$

is a finite **random set** where $A_j \subseteq \Omega, m_j := m(A_j)$, and $N := |\mathcal{S}| \le 2^n - 1$. Denote the **focal set** of $\mathcal{S}$ as the class $\mathcal{F}(\mathcal{S}) := \{A_j : m_j > 0\} \subseteq 2^\Omega$.

The **plausibility** and **belief** measures on $\forall A \subseteq \Omega$ are

$$\mathrm{Pl}(A) := \sum_{A_j \cap A \neq \emptyset} m_j, \qquad \mathrm{Bel}(A) := \sum_{A_j \subseteq A} m_j,$$

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Pl and Bel are generally normal, non-additive, dual fuzzy measures [10], with $\forall A \subseteq \Omega, \mathrm{Bel}(A) = 1 - \mathrm{Pl}(\overline{A})$.

Let $\mathcal{C} = \{A\} \subseteq 2^\Omega$ be a partition of Ω, and assume a special subset $A_0 \subseteq \Omega$ (not necessarily a member of the partition). Then $\mathbf{R}(A_0) := \{\underline{A}_0, \overline{A}_0\}$ is a **rough set** on Ω, where

$$\underline{A}_0 := \{A \in \mathcal{C} : A \subseteq A_0\},$$
$$\overline{A}_0 := \{A \in \mathcal{C} : A \cap A_0 \neq \emptyset\}.$$

2.2 Random Intervals, PBoxes, and Traces

Denote the class $\mathcal{D} := \{[a,b) \subseteq \mathbb{R} : a, b \in \mathbb{R}, a < b\}$ of half-open interval subsets of $\mathbb{R}$. In general, let $I := [a,b) \in \mathcal{D}$, and let $l(I) = a, u(I) = b$. Then a random interval, denoted $\mathcal{A}$, is a random set on $\Omega = \mathbb{R}$ for which $\mathcal{F}(\mathcal{A}) = \{I_j, 1 \leq j \leq N\} \subseteq \mathcal{D}$. Thus a random interval is a random left-closed interval subset of $\mathbb{R}$. An example is shown on the bottom of Fig. 1, with $N = 4$, $\mathcal{F}(\mathcal{A}) = \{[3.5,4), [1,2), [3,4), [2,3.5)\}$, support $\mathbf{U}(\mathcal{F}(\mathcal{A})) = [1,4)$, and m is as shown.

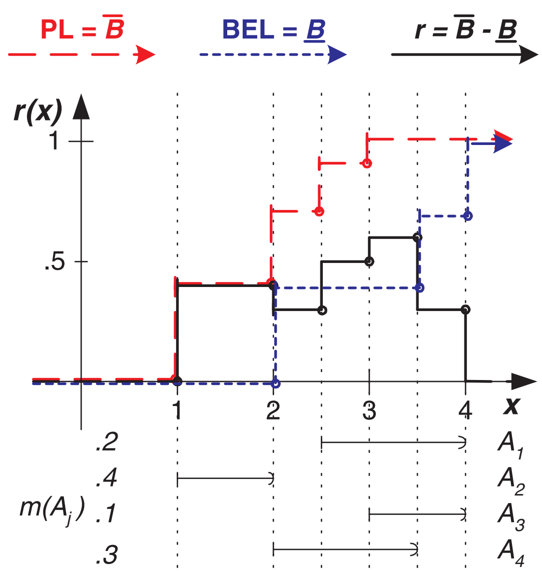

Figure 1. Example of a random interval.

Please note that while technically random intervals are best defined on half-open intervals, below we occasionally lapse into closed interval notation. The difference is not significant for the development in this paper.

A **PBox** [1] is a structure $\mathcal{B} := \langle \underline{B}, \overline{B} \rangle$, where $\underline{B}, \overline{B} \colon \mathbb{R} \mapsto [0,1]$,

$$\lim_{x \longrightarrow -\infty} B(x) \longrightarrow 0, \quad \lim_{x \longrightarrow \infty} B(x) \longrightarrow 1, \quad B \in \mathcal{B},$$

and $\underline{B}, \overline{B}$ are monotonic with $\underline{B} \leq \overline{B}$. $\underline{B}$ and $\overline{B}$ are interpreted as bounds on cumulative distribution functions (CDFs). In other words, given $\mathcal{B} = \langle \underline{B}, \overline{B} \rangle$, we can identify the set of all functions $\{F : \underline{B} \leq F \leq \overline{B}\}$ such that F is the CDF of some probability measures Pr on $\mathbb{R}$. Thus each PBox defines such a class of probability measures.

Given a random interval $\mathcal{A}$, then

$$\mathcal{B}(\mathcal{A}) := \langle \mathrm{BEL}, \mathrm{PL} \rangle \tag{1}$$

is a PBox, where BEL and PL are the "cumulative belief and plausibility distributions" $\mathrm{PL}, \mathrm{BEL} \colon \mathbb{R} \mapsto [0,1]$ originally defined by Yager [11]

$$\mathrm{BEL}(x) := \mathrm{Bel}((-\infty, x)), \quad \mathrm{PL}(x) := \mathrm{Pl}((-\infty, x)).$$

Given a random interval $\mathcal{A}$, define the function $r_{\mathcal{A}} \colon \mathbb{R} \mapsto [0,1]$ as the plausibilistic trace, or just **trace**, of $\mathcal{A}$, where $r_{\mathcal{A}}(x) := \mathrm{Pl}(\{x\})$. Given a random interval $\mathcal{A}$, then we also have that $r_{\mathcal{A}} = \mathrm{PL} - \mathrm{BEL}$, so that for a PBox derived from (1), we have

$$r_{\mathcal{A}} = \overline{B} - \underline{B}. \tag{2}$$

All the traces used in this paper have the form of a kind of possibility distribution called a fuzzy interval, specifically, a normal, convex, fuzzy subset of $\mathbb{R}$. See details elsewhere [3, 4, 5].

The PBox generated from the example random interval is shown in the top of Fig. 1. Since $\overline{B}$ and $\underline{B}$ partially overlap, the diagram is somewhat ambiguous on its far left and right portions, but note that

$$\overline{B}((-\infty, 1)) = 0, \quad \underline{B}((-\infty, 2,)) = 0,$$
$$\overline{B}([3, \infty)) = 1, \quad \underline{B}([3.5, \infty)) = 1.$$

The trace $r_{\mathcal{A}} = \overline{B} - \underline{B}$ is also shown.

So each random interval determines a PBox by (1), which in turn determines a trace by (2). But conversely, each trace determines an equivalence class of PBoxes, and each PBox an equivalence class of random intervals. In turn, each such equivalence class has a canonical member constructed by a standard mechanism. See elsewhere for details [5].

3 Stepwise Traces from Multi-Interval Elicitation

Our simulations are based on random intervals, and so we need methods for elicitation of these quantities from

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experts. But even explaining the mathematics from Sec. 2.2 is more work than most investigators want to invest in. Moreover, how to make such quantities as degrees of belief in overlapping intervals, or the BEL and PL curves, meaningful, is not at all clear. Indeed, investigators are much more willing to work with quantities from their application domain, or very simple mathematical structures such as the one-dimensional traces, which have similar appearance to familiar objects such as probability distributions.

Instead, we wish to elicit a simple structure such as an interval or a simple trace, and then derive the most well-justified random interval consistent with that structure. Elicitation of single intervals is relatively simple. Effectively, we ask of them about a particular quantity, "how high and low can it be?". Fig. 2 shows the situation for the answer "a and b". Denote this as

$$\mathrm{MIN}_0 = a, \quad \mathrm{MAX}_0 = b,$$

$$\mathrm{BOUNDS}_0 = [\mathrm{MIN}_0, \mathrm{MAX}_0] = [a, b).$$

We can then derive a random interval, which in this case is degenerate:

$$\mathcal{A}(\mathrm{BOUNDS}_0) = \{\langle [a, b), 1\rangle\}.$$

This is shown in Fig. 2, along with the PBox and trace.

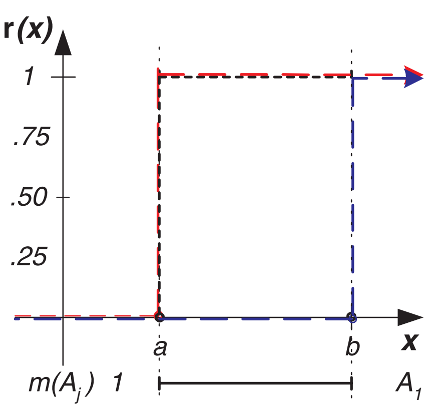

Figure 2. Random interval and PBox representation of "How high and low can it be?"

Our method for deriving a random interval begins with the elicitation of a structure we call a **multi-interval**. In addition to "how high" and "how low", we simply also elicit "how high and low might the min and max themselves be"? Let's say that the answer is "a and c, and d and b respectively", where we assume $c < d$ for simplicity. We can then denote this as:

$$\mathrm{MIN}_1 = [a, c], \quad \mathrm{MAX}_1 = [b, d],$$

$$\mathrm{BOUNDS}_1 = [\mathrm{MIN}_1, \mathrm{MAX}_1] = [[a, c], [b, d]],$$

where BOUNDS_1 is called a multi-interval. This is shown at the top of Fig. 3.

We then wish to derive a random interval representation of BOUNDS_1. Our method proceeds as follows.

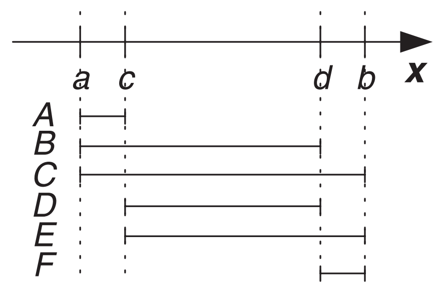

Figure 3. "How high and low can the high and low themselves go?"

- The bottom of Fig. 3 shows all possible intervals $\{A, B, \ldots, F\}$ derived from the four points $\{a, b, c, d\}$ of BOUNDS_1. We wish to construct a random interval from these intervals, in particular a collection of intervals which cover $[a, b]$.
- But, we're interested in considering all intervals derived from combining points from both MIN_1 and MAX_1 of BOUNDS_1. Thus intervals A and F are rejected, since they derive only from either MIN_1 or MAX_1 alone:

$$\begin{aligned} A = [a, c] &= [l(\mathrm{MIN}_1), u(\mathrm{MIN}_1)], \\ B = [d, b] &= [l(\mathrm{MAX}_1), u(\mathrm{MAX}_1)]. \end{aligned}$$

  This leaves the collection of viable intervals $V = \{B, C, D, E\}$.
- We are then interested in all collections of viable intervals which cover $[a, b]$:

$$\mathbf{C} := \{\mathcal{V} \subseteq V : \mathbf{U}(\mathcal{V}) = [a, b]\}.$$

  We have:

$$\begin{aligned} \mathbf{C} &= \{\{C\}, \{B, C\}, \{B, E\}, \{C, D\}, \{C, E\}, \\ &\quad \{B, C, D\}, \{B, C, E\}, \{B, D, E\}, \{C, D, E\}, \\ &\quad \{B, C, D, E\}\} \\ &= \{\mathcal{V}_i : 1 \leq i \leq 10\} \end{aligned}$$

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- For each such collection $\mathcal{V}_i \in \mathbf{C}$, we construct the random interval $\mathcal{A}(\mathcal{V}_i)$ by assigning equal probabilities $1/|\mathcal{V}_i|$ to each $I \in \mathcal{V}_i$. For example, for $\mathcal{V}_3 = \{B, E\}$, we have $\mathcal{A}(\mathcal{V}_3) = \{\langle B, .5\rangle, \langle E, .5\rangle\}$, which is shown in Fig. 4, along with its PBox $\mathcal{B}(\mathcal{A}(\mathcal{V}_3))$ and trace $r_{\mathcal{A}(\mathcal{V}_3)}$. In fact, we have that

$$\mathcal{B}(\mathcal{A}(\mathcal{V}_3)) = \mathcal{B}(\mathcal{A}(\mathcal{V}_4)) = \mathcal{B}(\mathcal{A}(\mathcal{V}_{10})),$$

$$r_{\mathcal{A}(\mathcal{V}_3))} = r_{\mathcal{A}(\mathcal{V}_4))} = r_{\mathcal{A}(\mathcal{V}_{10})},$$

and so also $\mathcal{V}_4$ and $\mathcal{V}_{10}$ are also shown in Fig. 4.

Fig. 4 shows all the distinct traces for the various $\mathcal{A}(\mathcal{V}_i)$. We observe the following:

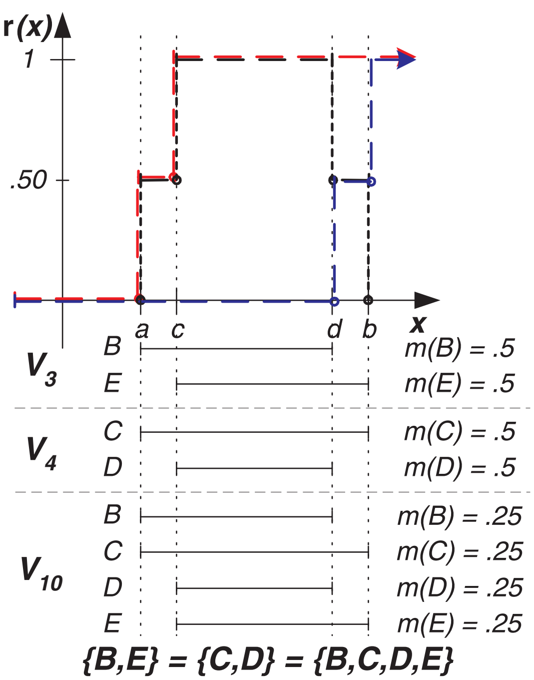

Figure 4. $\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_{10}$, and their common random interval and trace.

- $\{C\}$ by itself is clearly degenerate, equivalent to elicitation again of a single interval.
- Traces for $\{B, C\}$, and $\{C, E\}$ are marked by the fact that for either their left or right sides, the upper and lower values are not distinguished. For $\{C\}$, this is true for both.
- Of the remaining collections which have a real structure for both the left and right sides, three are symmetric with landmark values of $2/3, 1/3$, and $1/2$ for $\{B, C, E\}, \{B, D, E\}$, and the common structures from Fig. 4 respectively.
- Two others are asymmetric, with the values for the left side lower than the right for $\{C, D, E\}$, and *vice versa* for $\{B, C, D\}$.

4 Trace Elicitation of Random Intervals

So we now have a collection of random intervals $\mathcal{A}(\mathcal{V}_i), 1 \leq i \leq 10$, each of which is potentially consistent with an elicited multi-interval BOUNDS_1, and each of which yields a stepwise-constant PBox and trace. Each of these is blessed by the facts that they are directly supported by the observed data, the elicitation required is simple and natural, easy to communicate and interpret, and the resulting focal classes are very small, with $|\mathcal{F}|$ as low as 2, which greatly eases sampling effort in the simulation.

Given that a multi-interval by itself doesn't determine a unique random interval, and that more information can be taken from the investigator and integrated relatively easily, we can move on to consider eliciting a continuous, or at least piecewise continuous, trace. We have discussed [3] methods for constructing piecewise-continuous traces (in this case, possibility distributions) from stepwise-constant traces of finite random intervals. Denoting $R(\mathcal{A})$ for the set of all such traces consistent with a random interval $\mathcal{A}$, then Fig. 6 shows a number of members of $R(\mathcal{A}(\mathcal{V}_3))$. Note that $R(\mathcal{A})$ is equivalent to a rough set on $\mathbb{R}$, where

$$\mathcal{C} = \{(-\infty, a), [a, c), [c, d), [d, b), [b, \infty)\},$$

and A_0 is any interval $[x, y)$ where $x \in [a, c), y \in [d, b)$, yielding

$$\mathbf{R}(A_0) = \langle \underline{A}_0, \overline{A}_0 \rangle = \langle\ \{[c, d)\},\ \{[a, c), [c, d), [d, b)\}\ \rangle .$$

Proceeding is motivated by the following observations:

- If either $a = c, b = d$, or both, then the corresponding degenerate or partially degenerate case should be returned.
- For either endpoint, if no further information other than the difference between a and c (b and d) is available, then only a linear interpolation is justified, which is the "middle" trace for either end shown in Fig. 6.
- Consider then the $[a, c)$ endpoint, with analogous reasoning for $[b, d)$. What is of interest is whether

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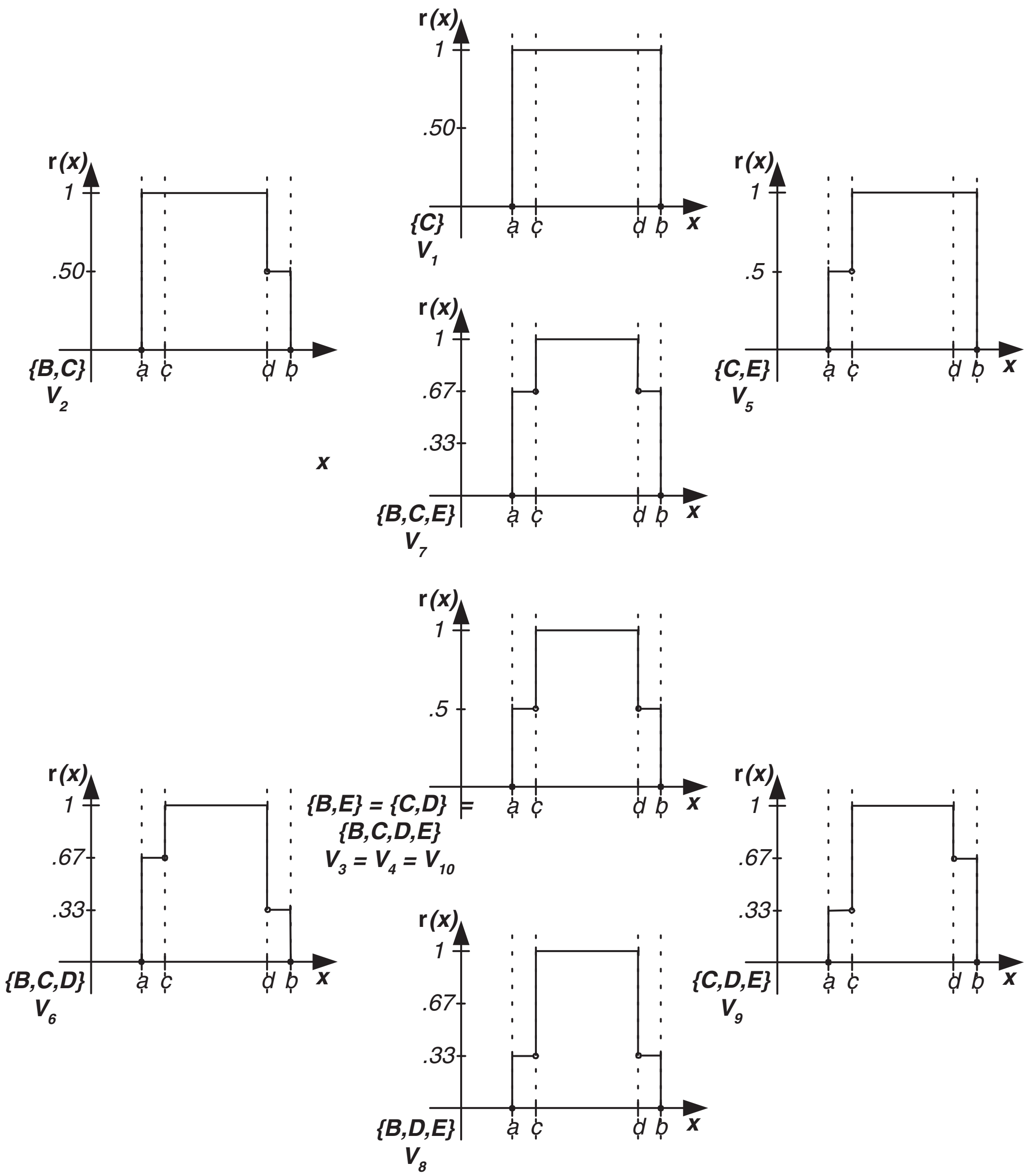

Figure 5. Random intervals, PBoxes, and traces consistent with the multi-interval BOUNDS_1.

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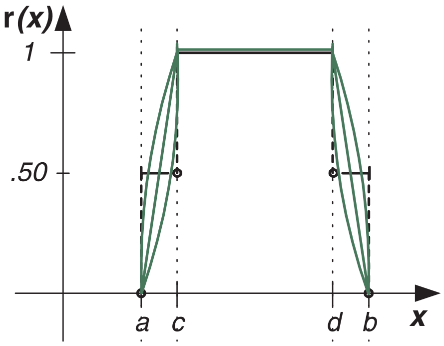

Figure 6. Possibility distributions consistent with two observed intervals.

		$[b,d)$		
		Convex	Concave	Unknown
	Convex	$\mathcal{V}_7$	$\mathcal{V}_6$	$\mathcal{V}_7$
$[a,c)$	Concave	$\mathcal{V}_9$	$\mathcal{V}_8$	$\mathcal{V}_8$
	Unknown	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{V}_{10}$

Table 1. Random interval assignment when both ends have structure.

the investigator has more information about how the uncertainty is distributed over the interval. In particular, is she more confident in the a than the c value? If so, then we draw and favor the convex trace shown, if not, then the concave trace.

The elicitation method then proceeds along the following algorithm to produce a random interval $\mathcal{A}$.

1. Elicit the upper and lower bound a, b.
2. Elicit the upper and lower bound of a: a, c.
3. If $a = c$, then
   - Elicit the upper and lower bound of b: b, d.
   - If $b = d$, then return $\mathcal{A}(\mathcal{V}_1)$
   - Otherwise return $\mathcal{A}(\mathcal{V}_2)$
4. If $b = d$, then return $\mathcal{A}(\mathcal{V}_5)$.
5. Elicit the convexity of $[a, c)$ and $[b, d)$.
6. Return $\mathcal{A}(\mathcal{V}_i)$ from Table 1.

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