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On Possibilistic Automata*

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Abstract. General automata are considered with respect to normalization over semirings. Possibilistic automata are defined as normal pessimistic fuzzy automata. Possibilistic automata are analogous to stochastic automata where stochastic $(+/\times)$ semirings are replaced by possibilistic $(\vee/\wedge)$ semirings; but where stochastic automata must be normal, fuzzy automata may be (resulting in possibilistic automata) or may not be (resulting in fuzzy automata proper). While possibilistic automata are direct generalizations of nondeterministic automata, stochastic automata are not. Some properties of possibilistic automata are considered, and the identity of possibilistic automata strongly consistent with stochastic automata is shown.

1 Introduction

Fuzzy automata were introduced and examined in the late 1960's in a series of papers by Santos [20], Santos and Wee [22], and Wee and Fu [25]. Their most commonly considered form is the **pessimistic max-min fuzzy automata**, a system $\widetilde{\mathcal{A}} := \langle Q, A, \mathbf{F}, \mu^0 \rangle$, where $Q := \{q_i\}$ is a finite set of **states**, $1 \leq i \leq N = |Q|$; $A := \{a_k\}$ is a finite **input alphabet**; $\mathbf{F} := \{F^k\}$ is a set of fuzzy binary relations on Q, denoted $F^k \mathrel{\tilde{\subset}} Q^2$, one for each input symbol a_k; and the **initial state** $\mu^0 \mathrel{\tilde{\subset}} Q$ is a fuzzy subset of Q. The **transition function** is then $\mu^n := F(n) \circ \mu^{n-1}$ at time $n \in \{1, 2, \ldots\}$, where $\circ$ is max-min composition and $F(n) := F(a_k(n)) = F^k \in \mathbf{F}$ is the binary relation for the symbol $a_k(n)$ input to $\widetilde{\mathcal{A}}$ at time n.

Some properties of fuzzy automata and languages were investigated [3, 18], and they were incorporated into a general theory of automata in the context of category theory [1]. Fuzzy automata and languages have received considerably less attention since the 1980's, but some progress continues [16, 17].

Gaines and Kohout [6] expanded the idea of fuzzy automata to include "possible automata", and in so doing described the basic concepts of **possibility theory** that would later be developed by Lotfi Zadeh [26] and others [5, 10]. Researchers have advanced possibility theory as a form of information theory which is distinct from, but related to, both probability theory and fuzzy theory. Therefore, an explicit definition of **possibilistic automata** is now required.

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2 Automata

2.1 General Automata

A general automaton can be defined [6, 21] as a process over an ordered, commutative semiring $\mathcal{V} := \langle V, \oplus, \otimes, 0, 1, \leq \rangle$, where $V := \{v\}$; $\oplus$ and $\otimes$ are general operators over V; $0, 1 \in V$ are the identities for $\oplus$ and $\otimes$ respectively; usually $V \subseteq \mathbb{R}$ with $\{0, 1\} \subseteq V \subseteq [0, 1]$; and $\leq$ is a (usually complete) order on V. For convenience, denote $\mathcal{V} = \langle V, \oplus, \otimes \rangle$.

Definition 1 General Finite Automaton. A system $\mathcal{A} := \left\langle Q, A, \mathcal{V}, \sigma, \phi^0 \right\rangle$ is a general automaton where:

- $\sigma: Q \times A \times Q \mapsto V$ is the **transition function**;
- $\phi^0: Q \mapsto V$ is the **initial state**; and
- $\phi^n: Q \mapsto V$ is the **state function** for time n with

$$\phi^n(q_i) := \bigoplus_{q_j \in Q} \sigma(q_i, a_k(n), q_j) \otimes \phi^{n-1}(q_j) \ . \tag{1}$$

$\widetilde{\mathcal{A}}$ results when: $\mathcal{V} = \langle [0, 1], \vee, \wedge \rangle$; $\vee$ and $\wedge$ are the maximum and minimum operators; $\mu^n = \phi^n$; and $F(n) = \sigma(\cdot, a_k(n), \cdot)$ is taken to be an $N \times N$ **transition matrix**. Other classes of fuzzy automata include **max-product fuzzy automata** for $\mathcal{V} = \langle [0, 1], \vee, \times \rangle$ and **optimistic min-max fuzzy automata** for $\mathcal{V} = \langle [0, 1], \wedge, \vee \rangle$ and the ordering $\leq$ on $[0, 1]$ reversed to $\geq$.

2.2 Normal General Automata

Stochastic automata result for $\mathcal{V} = \langle [0, 1], +, \times \rangle$. Letting i and j be general indices over Q, and denoting $p_i^n := \phi^n(q_i)$ and $p_{ij}^k = p(q_i | a_k, q_j) := \sigma(q_i, a_k, q_j)$ as the state and transition probabilities respectively, then the well-known stochastic transition function is

$$p_i^n = \sum_{j=1}^{N} p_{ij}^k \, p_j^{n-1} \ . \tag{2}$$

However, $v, v' \in [0, 1] \not\to v + v' \in [0, 1]$. In general $p_i^n \in [0, 2] \neq V$. To guarantee the closure of state transitions the following **stochastic normalization conditions** are required:

$$\sum_{i=1}^{N} p_i^0 = 1, \qquad \underset{a_k \in A}{\forall} \ \overset{N}{\underset{j=1}{\forall}} \ \sum_{i=1}^{N} p_{ij}^k = 1 \ . \tag{3}$$

On generalization of (3) to $\mathcal{V}$ the following **general normalization conditions** are obtained.

Definition 2 State Normalization. A state function $\phi: Q \mapsto V$ is normal when

$$\bigoplus_{q_i \in Q} \phi(q_i) = 1 \ . \tag{4}$$

Definition 3 Automaton Normalization. An automaton $\mathcal{A}$ is normal when ϕ^0 is normal and

$$\underset{a_k \in A}{\forall} \; \underset{q_j \in Q}{\forall} \; \bigoplus_{q_i \in Q} \sigma(q_i, a_k, q_j) = 1 \ . \tag{5}$$

If $\mathcal{A}$ is normal, then state normalization holds in general [6].

Theorem 4. *If $\mathcal{A}$ is normal, then $\forall n \geq 0, \phi^n$ is normal.*

Proof. Proof by induction. (1) Since $\mathcal{A}$ is normal, ϕ^0 is normal. (2) For $n \geq 1$, assume ϕ^{n-1} is normal. Fix a_k and n, and denote

$$\phi'_i := \phi^n(q_i), \quad \phi_j := \phi^{n-1}(q_j), \quad \sigma_{ij} := \sigma(q_i, a_k, q_j), \quad vv' := v \otimes v' \ . \tag{6}$$

Then in virtue of the commutivity of $\oplus$ and $\otimes$, the distributivity of $\otimes$ over $\oplus$, the normality of $\mathcal{A}$, the identity of 1 for $\otimes$, and the normality of ϕ (respectively), we have

$$\begin{aligned} \bigoplus_i \phi'_i &= \bigoplus_i \bigoplus_j \sigma_{ij}\phi_j = \bigoplus_j \bigoplus_i \phi_j \sigma_{ij} \\ &= \bigoplus_j \phi_j \bigoplus_i \sigma_{ij} = \bigoplus_j \phi_j 1 = \bigoplus_j \phi_j = 1 \ . \end{aligned} \tag{7}$$

□

When, and only when, normalization holds, then the final classes of classical automata can be recovered for discrete V: **nondeterministic automata** for $\mathcal{V} = \langle \{0,1\}, \vee, \wedge \rangle$ and **deterministic automata** for $\mathcal{V} = \langle \{0,1\}, +, \times \rangle$ [6].

3 Possibility Theory

Possibility theory is a new form of mathematical information theory. It is related to, but distinct from, both probability and fuzzy set theory, and arises independently in both Dempster-Shafer evidence theory and fuzzy set theory [5, 7, 14].

3.1 Mathematical Possibility Theory

A set function $m\colon 2^Q \mapsto [0,1]$ is an **evidence function** (otherwise known as a **basic probability assignment**) when $m(\emptyset) = 0$ and $\sum_{G \subseteq Q} m(G) = 1$. The **random set** generated by an evidence function is

$$\mathcal{S} := \{ \langle G_j, m_j \rangle : m_j > 0 \} \ , \tag{8}$$

where $\langle \cdot \rangle$ is a vector, $G_j \subseteq Q, m_j := m(G_j)$, and $1 \leq j \leq |\mathcal{S}| \leq 2^n - 1$. $\mathcal{S}$ has **focal set** $\mathcal{F} := \{G_j : m_j > 0\}$.

The dual **belief** and **plausibility** measures are

$$\forall G \subseteq Q, \quad \mathrm{Bel}(G) := \sum_{G_j \subseteq G} m_j \quad \mathrm{Pl}(G) := \sum_{G_j \cap G \neq \emptyset} m_j \tag{9}$$

respectively. Both are **fuzzy measures** [24]. The **plausibility assignment** (otherwise known as the **contour function**, **falling shadow**, or **one-point coverage function**) of $\mathcal{S}$ is

$$\mathbf{Pl} := \langle \mathrm{Pl}(\{q_i\}) \rangle = \langle \mathrm{Pl}_i \rangle, \qquad \mathrm{Pl}_i := \sum_{G_j \ni q_i} m_j \ . \tag{10}$$

When $\forall G_j \in \mathcal{F}, |G_j| = 1$, then $\mathcal{S}$ is **specific**, and $\forall G \subseteq Q, \mathrm{Pr}(G) := \mathrm{Bel}(G) = \mathrm{Pl}(G)$ is a **probability measure** with **probability distribution** $p = \langle p_i \rangle := \mathbf{Pl}$, additive normalization $\sum_i p_i = 1$, and operator $\mathrm{Pr}(G) = \sum_{q_i \in G} p_i$.

$\mathcal{S}$ is **consonant** ($\mathcal{F}$ is a **nest**) when (without loss of generality for ordering, and letting $G_0 = \emptyset$) $G_{j-1} \subseteq G_j$. Now $\Pi(G) := \mathrm{Pl}(G)$ is a **possibility measure** with dual **necessity measure** $\eta(G) := \mathrm{Bel}(G)$. As Pr is additive, so Π is **maximal** in the sense that

$$\forall G_1, G_2 \in \mathcal{F}, \quad \Pi(G_1 \cup G_2) = \Pi(G_1) \vee \Pi(G_2) \ . \tag{11}$$

Denoting $G_i := \{q_1, q_2, \ldots, q_i\}$, and assuming that $\mathcal{F}$ is complete ($\forall q_i \in Q, \exists G_i \in \mathcal{F}$), then $\pi = \langle \pi_i \rangle := \mathbf{Pl}$ is a **possibility distribution** with maximal normalization and operator

$$\bigvee_{i=1}^{n} \pi_i = \pi_1 = 1, \qquad \Pi(G) = \bigvee_{q_i \in G} \pi_i \ . \tag{12}$$

3.2 Possibility and Fuzzy Theory

It is clear that π is a normal fuzzy set, and indeed, Zadeh defined possibility strictly in terms of fuzzy sets in his original introduction of possibility theory [26]. But the relation between possibility and fuzziness is actually a bit more involved.

Let the cardinality of a fuzzy set $\phi \mathrel{\tilde{\subset}} Q$ with membership function $\mu_\phi : Q \mapsto [0,1]$ be $|\phi| := \sum_i \mu_\phi(q_i)$. Kampè de Fériet has shown [12] that for countable Q, and for some random set $\mathcal{S}$ and fuzzy set ϕ taken as a vector, that:

- $|\phi| \geq 1$ iff ϕ can be taken as a plausibility assignment $\mathbf{Pl}$;
- similarly, $|\phi| \leq 1$ iff ϕ can be taken as a "belief assignment" $\mathbf{Bel} := \langle \mathrm{Bel}(\{q_i\}) \rangle$; and finally
- $|\phi| = 1$ iff ϕ can be taken as a probability distribution p.

In the first two cases, only a mapping back to an equivalence class of random sets is guaranteed, while in the last case each additive fuzzy set maps to a unique specific (probabilistic) random set. Finally, if $\bigvee_i \mu_\phi(q_i) = 1$ then the first case holds, but also $\mathbf{Pl} = \pi$ is a possibility distribution mapping back to an equivalence class of consonant (possibilistic) random sets [7].

3.3 Interpretations of Possibility

While possibility theory has almost invariably been related directly to fuzzy set theory, probability theory has been regarded as independent from it. This confusion may be a result of the fact that possibility theory is a very weak representation of uncertainty, whereas probability makes very strong requirements.

Furthermore, the wedding of possibility theory to fuzzy sets has relegated possibility to be interpreted strictly in accordance with fuzzy semantics. Since the founding of fuzzy theory by Zadeh in the 1960's, fuzziness has been interpreted almost exclusively as a *psychological* form of uncertainty, expressed in natural language (or "linguistic variables"), and measured by the subjective evaluations of human subjects.

A number of researchers are developing possibility theory into a complete, alternative information theory. Unique measures of information, analogs of stochastic entropy, are being developed [13], along with a possibilistic semantics of *natural* systems which is independent of *both* fuzzy sets and probability [9, 10, 11].

4 Possibilistic Automata

4.1 Definition

Let $\boldsymbol{\Pi}(Q) := \{\pi\}$ be the set of all possibility distributions on Q so that $\pi\colon Q \mapsto [0,1]$ and

$$\forall \pi \in \boldsymbol{\Pi}(Q), \qquad \bigvee_{q_i \in Q} \pi(q_i) = 1 \tag{13}$$

is the normalization requirement.

Definition 5 Possibilistic Automaton. A system $\mathcal{A}^\pi := \left\langle Q, A, \pi, \pi^0 \right\rangle$ is a possibilistic automaton if $\left\langle Q, A, \langle [0,1], \vee, \wedge \rangle, \pi, \pi^0 \right\rangle$ is a normal general finite automaton, $\pi \in \boldsymbol{\Pi}(Q|A \times Q)$ is the **transition function**, and $\pi^0 \in \boldsymbol{\Pi}(Q)$ is the **initial state**.

$\pi(q_i|a_k, q_j)$ is the **conditional possibility**[4] of transiting to state q_i from state q_j given an input symbol a_k. The normalization condition (5) becomes

$$\forall a_k, q_j, \qquad \bigvee_{q_i \in Q} \pi(q_i|a_k, q_j) = 1 \ , \tag{14}$$

which states that no matter what the current state, there must be at least one state to which it is completely possible to transit.

The state function is

$$\pi^n = \langle \pi_i^n \rangle = \boldsymbol{\pi}(n) \circ \pi^{n-1} \ , \tag{15}$$

[4] An exact mathematical definition or interpretation of conditional possibility is still somewhat controversial [8, 19].

where: $\pi_i^n := \pi^n(q_i)$ is the possibility of being in state q_i at time n; and $\boldsymbol{\pi}(n) = [\boldsymbol{\pi}(n)_{ij}] := \pi(\cdot|a_k(n), \cdot)$ is the $N \times N$ **transition matrix** at time n. π^n is a column vector; i indexes the rows of π^n and $\boldsymbol{\pi}(n)$; and j indexes the columns of $\boldsymbol{\pi}(n)$. In general

$$\pi_i^n = \bigvee_{j=1}^{N} \boldsymbol{\pi}(n)_{ij} \wedge \pi_j^{n-1} \quad . \tag{16}$$

$\mathcal{A}^\pi$ is a special case of $\widetilde{\mathcal{A}}$, under the assignments $F(n) = \boldsymbol{\pi}(n)$ and $\phi^n = \pi^n$. Santos identified $\mathcal{A}^\pi$ as a **restricted fuzzy automaton** [20].

4.2 An Example

For a simple example, let $\pi^0 = \langle 0.3, 1.0, 0.6 \rangle^T$ and let $\boldsymbol{\pi}$ be fixed at

$$\boldsymbol{\pi} = \begin{bmatrix} 0.1 & 0.0 & 0.4 \\ 1.0 & 0.5 & 1.0 \\ 0.2 & 1.0 & 0.7 \end{bmatrix} \quad . \tag{17}$$

π^0 is normal, as is $\boldsymbol{\pi}$, with a 1 in each column. The state vector changes as follows:

n	0	1	2	3	4
π^n	0.3	0.4	0.4	0.4	0.4
	1.0	0.6	1.0	0.7	1.0
	0.6	1.0	0.7	1.0	0.7

Since $\pi^2 = \pi^4$, the cycle will repeat. This is completely in keeping with the periodic behavior of fuzzy matrix composition [23]. By Thm. 4, each π^n is normal, although the normalizing element rotates among the π_i^n. Further properties are considered in Sec. 5.

4.3 Discussion

It is significant to note that not all fuzzy relations which are normal in the sense of fuzzy sets are normal in the sense of possibilistic transition matrices (while the converse *is* true). Unlike (14), fuzzy relation normalization [14] only requires that

$$\forall a_k, \bigvee_{q_i, q_j \in Q} \pi(q_i|a_k, q_j) = 1 \quad ; \tag{18}$$

that is, that there is *some* unitary element in each $\boldsymbol{\pi}(n)$, not that there is some unitary element in *each column* of each $\boldsymbol{\pi}(n)$.

When $\oplus = \vee$, then we call $\mathcal{V}$ a **possibilistic semiring**. Contrary to stochastic automata, $v, v' \in [0,1] \to v \oplus v' \in [0,1]$. Therefore for fuzzy automata with possibilistic semirings, $\phi^n(q_i) \in [0,1]$ *whether $\mathcal{A}$ is normal or not.* By Thm. 4, π^n will be normal in general. But while this is also true for stochastic automata, it is not the case for all fuzzy automata.

In Sec. 2.2 it was shown that both deterministic and stochastic automata are based on the same algebraic structure which we will call a **stochastic semiring** $\langle \oplus, \otimes \rangle = \langle +, \times \rangle$. In fact, stochastic automata are direct generalizations of deterministic automata when $V = \{0,1\}$ is replaced by its superset $V = [0,1]$. Similarly, fuzzy automata are direct generalizations of nondeterministic automata for $V = [0,1]$ with possibilistic semirings. Finally, possibilistic automata result when normalization is imposed on fuzzy automata, while normalization is automatic for both stochastic and deterministic automata. Since for nondeterministic automata

$$\forall i, j \quad \pi_i^n, \boldsymbol{\pi}(n)_{i,j} \in \{0,1\} \ , \tag{19}$$

therefore all nondeterministic automata are also possibilistic automata, and are identified as **crisp possibilistic automata**.

Thus proper (non-crisp) possibilistic automata are also direct generalizations of nondeterministic automata. And it is significant to note that stochastic automata are *not* generalizations of nondeterministic automata, despite years of commitment to them as the sole embodiment of machines with quantified uncertainty. These relations among these classes of automata are summarized in Fig. 1.

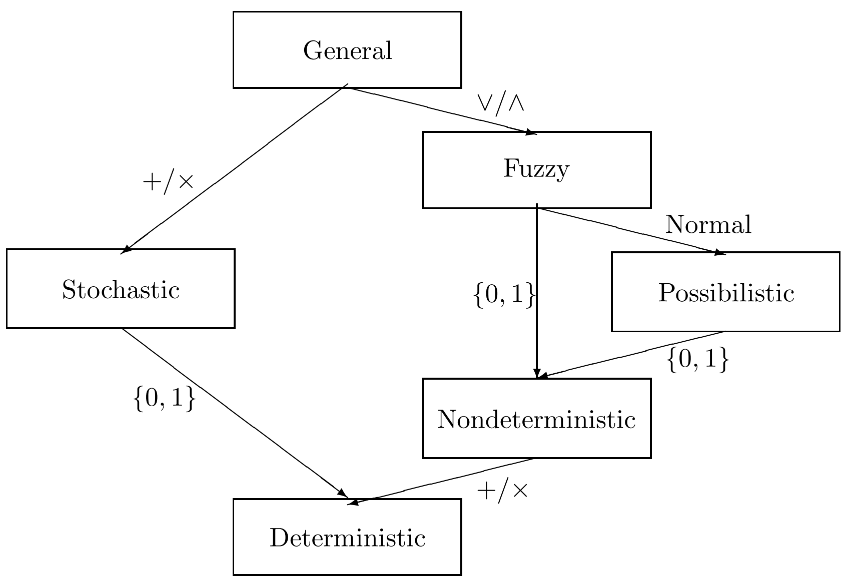

Fig. 1. Classes of automata.

5 Properties

The actions of possibilistic automata are dependent on a maximally normalized initial state vector π^0 and conditional transition matrices $\boldsymbol{\pi}(n)$, and thus on the presence of 1's in $\boldsymbol{\pi}$. In investigating their properties, the following lemma will be useful.

Lemma 6. *Let $R := [r_{ij}], S := [s_{ij}], T := [t_{ij}]$ be $N \times N$ matrices with $0 \leq r_{ij}, s_{ij}, t_{ij} \leq 1$, and let $R \circ S = T$. If $\exists r_{i^*j^*} = 1$, then $\forall j, t_{i^*j} \geq s_{j^*j}$.*

Proof. Assume $r_{i^*j^*} = 1$. Then $\forall j$,

$$\begin{aligned} t_{i^*j} &= \bigvee_{k=1}^{N} r_{i^*k} \wedge s_{kj} \\ &= \left(\bigvee_{\substack{1 \leq k \leq N \\ k \neq j^*}} r_{i^*k} \wedge s_{kj} \right) \vee (r_{i^*j^*} \wedge s_{j^*j}) = \left(\bigvee_{\substack{1 \leq k \leq N \\ k \neq j^*}} r_{i^*k} \wedge s_{kj} \right) \vee s_{j^*j} \\ &\geq s_{j^*j} \ . \end{aligned} \tag{20}$$

□

In the sequel, assume a possibilistic automaton $\mathcal{A}^\pi$ with fixed input symbol $a_n(k)$ and single transition matrix $\boldsymbol{\pi} := [\pi_{ij}]$. Given a state vector π, then the next-state vector is $\pi' := \boldsymbol{\pi} \circ \pi$. Where appropriate, the nth step state vector will be denoted π^n.

Whereas each column of $\boldsymbol{\pi}$ is guaranteed to have at least one unitary value, this is not necessarily the case for the rows. For each row i, let $C(i) := \{c_k^i\}$ be the set of indices (if any) for which

$$\pi_{i,c_k^i} = 1, \quad 1 \leq k \leq |C(i)| \leq N \ , \tag{21}$$

and let $c(i)$ be the unique such index in each row i, if it exists. The column positions of the unitary elements in a row determine which elements of the state vector will provide lower bounds for the corresponding element of the next-state vector. Where there are multiple unitary elements in a row, then the bound will be the maximum of all corresponding state values.

Theorem 7. *$\forall i$, if $C(i) \neq \emptyset$, then*

$$\pi'_i \geq \bigvee_{c_k^i \in C(i)} \pi_{c_k^i} \ . \tag{22}$$

Proof. $\forall i, \forall c_k^i \in C(i), \pi_{i,c_k^i} = 1$. Under the assignments $R_{N\times N} = \boldsymbol{\pi}, S_{N\times 1} = \pi, T_{N\times 1} = \pi'$, then by Lem. 6, $\forall i, \forall c_k^i \in C(i), \pi'_i \geq \pi_{c_k^i}$, and so the conclusion follows. □

Corollary 8. *$\forall i$, if $c(i)$ exists then $\pi'_i \geq \pi_{c(i)}$.*

Proof. Follows directly from Thm. 7 under the assumption that $C(i) = \{c(i)\}$. □

If each row of $\boldsymbol{\pi}$ has a unique unitary element, then $c(i)$ exists and is an injective function creating a cyclic group permuting the elements of Q through at most N steps. Similarly, the bounds of the state vector will cyclically permute through a maximum of N elements of π.

Theorem 9. *If $\forall i, c(i)$ exists, then $\forall i, \exists e(i), 1 \leq e(i) \leq N$ such that $\pi_i^{e(i)} \geq \pi_i^0$.*

Proof. Let $\lceil x \rceil$ be any value such that $\lceil x \rceil \in [x, 1]$, and let $\pi^0 = \langle \pi_1, \pi_2, \ldots, \pi_N \rangle^T$. By Cor. 8, the action of the automaton in the first step affects the transformation

$$\pi^0 \mapsto \pi^1 = \left\langle \lceil \pi_{c(1)} \rceil, \lceil \pi_{c(2)} \rceil, \ldots, \lceil \pi_{c(N)} \rceil \right\rangle^T \quad . \tag{23}$$

The next step of the automaton affects the further transformation

$$\pi^1 \mapsto \pi^2 = \left\langle \lceil \pi_{c^2(1)} \rceil, \lceil \pi_{c^2(2)} \rceil, \ldots, \lceil \pi_{c^2(N)} \rceil \right\rangle^T \quad . \tag{24}$$

In general,

$$\pi^n = \left\langle \lceil \pi_{c^n(1)} \rceil, \lceil \pi_{c^n(2)} \rceil, \ldots, \lceil \pi_{c^n(N)} \rceil \right\rangle^T \quad . \tag{25}$$

Since $\forall i, \exists e(i) \leq N, c^{e(i)}(i) = i$, therefore

$$\forall i, \quad \pi_i^{e(i)} = \lceil \pi_{c^{e(i)}(i)} \rceil = \lceil \pi_i \rceil \geq \pi_i \ . \tag{26}$$

□

The placement of unitary values on the diagonal of $\boldsymbol{\pi}$ guarantee a monotonic increase of state vector values.

Corollary 10. *If $\exists i, c(i) = i$, then $\pi'_i \geq \pi_i$.*

Proof. Follows immediately from Cor. 8. □

Finally, if $\boldsymbol{\pi}$ has unitary values on all diagonal elements, then a result of Pedrycz [18] is recovered.

Corollary 11. *If $\forall i, c(i) = i$, then $\pi' \geq \pi$.*

Proof. Follows immediately from the application of Cor. 10 to all columns. □

6 Consistency with Stochastic Concepts of Possibility

While the mathematical syntax of possibility theory has become relatively well established, the semantics of possibility values has not been developed in its own right. Instead, possibility values have only been interpreted in the context of either fuzzy sets or probability. As mentioned in Sec. 3.3, possibility values are typically interpreted as membership grades of a fuzzy set. Possibility values are then determined from a given fuzzy set, which is itself usually determined by some subjective valuation procedure.

6.1 Strong Probability–Possibility Consistency

The other major set of methods developed to date derive possibility values by converting a given probability or frequency distribution. Such conversion methods have been guided by a principle of **probability-possibility consistency**, briefly stated by Delgado and Moral as "the intuitive idea according to which as an event is more probable, then it is more possible [2]," and summarized most generally by the formula

$$\forall G \subseteq Q, \quad \mathrm{Pr}(G) \leq \mathit{\Pi}(G) \ . \tag{27}$$

In Zadeh's initial introduction of possibility theory [26] he also proposed a quantitative measure of this consistency

$$\gamma(\pi, p) := \sum_i \pi_i p_i \in [0, 1] \ , \tag{28}$$

such that $\gamma(\pi, p) = 0$ indicates complete inconsistency and $\gamma(\pi, p) = 1$ complete consistency. This measure has been generalized by Delgado and Moral [2].

Given a probability distribution $p = \langle p_i \rangle, \sum_i p_i = 1$, then the most traditional conversion formula is "maximum normalization" according to the formula

$$\pi_i = \frac{p_i}{\bigvee_{i=1}^{N} p_i} \ . \tag{29}$$

Other methods have also been suggested [4, 15].

However, recently Joslyn [10] has proposed a reconsideration of the relation between probability and possibility based on *semantic* criteria, and in particular the natural meaning of probability and possibility. This view incorporates both the modal sense of possibility and necessity, which accommodate only crisp possibility values ($\pi_i \in \{0, 1\}$); and a "stochastic" interpretation of possibility, in which an event with *any* positive probability must be *completely* possible.

The following **strong compatibility principle** [10] results,

$$\forall G \subseteq Q, \quad \mathrm{Pr}(G) > 0 \leftrightarrow \mathit{\Pi}(G) = 1 \tag{30}$$

which implies

$$\forall G \subseteq Q, \quad \mathrm{Pr}(G) = 0 \leftrightarrow \mathit{\Pi}(G) < 1 \ . \tag{31}$$

According to this principle, positive probability and complete possibility are synonymous, and thus the domains of applicability of probability and possibility should be considered as essentially distinct. Furthermore, when (30) holds, then $\gamma(\pi, p) = 1$; (27) holds; and the distributions are also strongly compatible [10]:

$$\forall q_i, \qquad p_i > 0 \leftrightarrow \pi_i = 1, \quad p_i = 0 \leftrightarrow \pi_i < 1 \ . \tag{32}$$

6.2 Consistency and Automaton Action

Strong compatibility of (30) is consistent with the operation of possibilistic automata. In particular, the operation of a stochastic automaton which is then converted to a possibilistic automaton according to (30) is equivalent to one which is converted first, and then operated by possibilistic formula.

Assume matrices R and S with $r_{ij}, s_{ij} \in [0,1]$ (no normalization of any kind assumed). Let $\overline{R} := [\overline{r_{ij}}]$, where

$$\overline{r_{ij}} := \begin{cases} 1, \, r_{ij} > 0 \\ 0, \, r_{ij} = 0 \end{cases} , \tag{33}$$

(and similarly for $\overline{S}$), and let RS denote matrix composition under a stochastic semiring (i.e., standard matrix multiplication).

Theorem 12. *$\overline{RS} = \overline{R} \circ \overline{S}$.*

Proof. Let $T = [t_{ij}] := RS$, so that $\overline{RS} = \overline{T} = [\,\overline{t_{ij}}\,]$. Also let $U = [u_{ij}] := \overline{R} \circ \overline{S}$. We need to prove that $\forall i, j, \overline{t_{ij}} = u_{ij}$. First, we have $t_{ij} = \sum_{k=1}^{n} r_{ik} s_{kj}$, so that

$$\overline{t_{ij}} = \begin{cases} 0, \quad \underset{1 \le k \le m}{\forall} \quad r_{ik} = 0 \text{ or } s_{kj} = 0 \\ 1, \text{ Otherwise} \end{cases} . \tag{34}$$

Then

$$\begin{aligned} u_{ij} &= \bigvee_{k=1}^{m} (\overline{r_{ik}} \wedge \overline{s_{ik}}) = \begin{cases} 0, \quad \underset{1 \le k \le m}{\forall} \quad \overline{r_{ik}} = 0 \text{ or } \overline{s_{ik}} = 0 \\ 1, \text{ Otherwise} \end{cases} \\ &= \begin{cases} 0, \quad \underset{1 \le k \le m}{\forall} \quad r_{ik} = 0 \text{ or } s_{ik} = 0 \\ 1, \text{ Otherwise} \end{cases} \\ &= \overline{t_{ij}} \ . \end{aligned} \tag{35}$$

□

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