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Aggregation and Completion in Probability and Possibility Theory*

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1 Introduction

Possibility theory [1] is an alternative information theory to that based on **probability**. Although possibility theory is logically independent of probability theory, they are related: both arise in **Dempster-Shafer evidence theory** as **fuzzy measures** defined on **random sets**; and their distributions are both **fuzzy sets**. So possibility theory is a component of a broader **Generalized Information Theory** (GIT), which includes all of these fields [9].

Possibility theory was originally developed by Zadeh in the context of fuzzy systems theory [12]. Possibility distributions have traditionally been interpreted *strictly* as fuzzy sets, and were thus related to the kinds of cognitive modeling that fuzzy sets are usually used for.

More recently, possibility theory is being developed on the alternative basis of the interaction between **random set theory** [8] and **evidence theory** [10], and thus independently of both fuzzy sets and probability. In particular, the author is developing [7] the mathematics and semantics of possibility theory based on consistent random sets [3, 4]. These methods include objective **possibilistic measurement** procedures based on **set-statistics** [2, 5] and **possibilistic processes** such as **possibilistic automata** [6] — generalizations of nondeterministic processes whose non-additive weights adhere to the laws of mathematical possibility theory.

In this paper some new ideas about the mathematical relations between probability and possibility are presented in the context of random set theory. Although some of these results are already known, I believe that the concepts of complete random sets and distributions, and numerical and structural aggregation functions on them, provide a broad, consistent context not only in which to place probability and possibility, but also to consider new forms of measures and distributions which have yet to be studied.

Below is a synoptic summary of the mathematical ideas to be presented in the full paper.

2 General Random Sets and Distributions

Assume a finite universe $\Omega := \{\omega_i\}, 1 \leq i \leq n$. The function $\nu : 2^\Omega \mapsto [0,1]$ is a **fuzzy measure** [11] if $\nu(\emptyset) = 0$ and

$$\forall A, B \subseteq \Omega, \quad A \subseteq B \rightarrow \nu(A) \leq \nu(B).$$

Then

$$q_\nu : \Omega \mapsto [0,1], \qquad q_\nu(\omega) := \nu(\{\omega\}),$$

is a **distribution** of ν if there exists a distribution operator function $\oplus : [0,1]^2 \mapsto [0,1]$ where $\langle [0,1], \oplus, 0 \rangle$ is an Abelian monoid ($\oplus$ is a commutative, associative, operator with identity 0) and, in operator notation,

$$\forall A \subseteq \Omega, \quad \bigoplus_{\omega \in A} q_\nu(\omega) = \nu(A).$$

The function $m : 2^\Omega \mapsto [0,1]$ is an **evidence function** (otherwise known as a **basic probability assignment**) when $m(\emptyset) = 0$ and

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$\sum_{A \subseteq \Omega} m(A) = 1$. Denote a **random set** generated from an evidence function as

$$\mathcal{S} := \{\langle A_j, m_j \rangle : m_j > 0\},$$

where $\langle \cdot \rangle$ is a vector, $A_j \subseteq \Omega, m_j := m(A_j)$, and

$$1 \leq j \leq N := |\mathcal{S}| \leq 2^n - 1.$$

Denote the **focal set** of $\mathcal{S}$ as $\mathcal{F} := \{A_j : m_j > 0\}$.

The **plausibility** and **belief** measures on $\forall A \subseteq \Omega$ are

$$\mathrm{Pl}(A) := \sum_{A_j \cap A \neq \emptyset} m_j, \qquad \mathrm{Bel}(A) := \sum_{A_j \subseteq A} m_j.$$

These are generally non-additive fuzzy measures, and are dual, in that $\forall A \subseteq \Omega, \mathrm{Bel}(A) = 1 - \mathrm{Pl}(\overline{A})$. In general only plausibility will be considered below. The **plausibility assignment** (otherwise known as the **one-point coverage function**) of $\mathcal{S}$ is $\vec{\mathrm{Pl}} = \langle \mathrm{Pl}_i \rangle := \langle \mathrm{Pl}(\{\omega_i\}) \rangle$, where

$$\mathrm{Pl}_i := \sum_{A_j \ni \omega_i} m_j.$$

If a distribution operator $\oplus$ exists, then clearly $\vec{\mathrm{Pl}}$ is the vector representation of $q = q_{\mathrm{Pl}}$, where $q_i = \mathrm{Pl}_i$.

Corollary 1 If q is a distribution of Pl on a random set $\mathcal{S}$, then

$$\bigoplus_{\omega_i \in \Omega} q_i = 1.$$

Definition 2 Given a random set $\mathcal{S}$ with a distribution q, then a function $g_q : \mathcal{F} \mapsto \Omega$ is a **structural aggregation function** if it is one to one.

Definition 3 Given a random set $\mathcal{S}$ with evidence function m, distribution q, and structural aggregation function g_q, then a function $h_{m,q} : [0,1] \mapsto [0,1]$ is a **numerical aggregation function** if $h_{m,q}(m(A_j)) = q(g_q(A_j))$.

Corollary 4 $|\mathcal{S}| = N \leq |\Omega| = n$.

Corollary 5 Under the relabeling

$$\omega_j := g_q(A_j), \qquad q_j := q(\omega_j) = q(g_q(A_j)), \quad (6)$$

then $h_{m,q}(m_j) = q_j$.

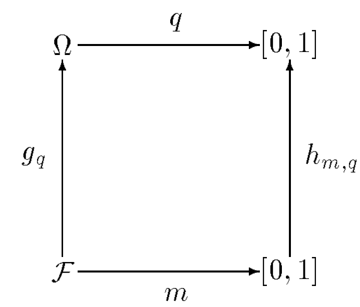

Figure 1: Relations among random sets and their distributions and aggregation functions.

These relations are diagrammed in Fig. 1.

Definition 7 Given a random set $\mathcal{S}$ with a structural aggregation function g_q, then $\mathcal{S}$ and its distribution q are **complete** if $N = |\mathcal{S}| = n = |\Omega|$.

Proposition 8 If q is complete, then g_q and $h_{m,q}$ are onto, and thus bijections, with

$$g_q^{-1}(\omega_j) = A_j, \qquad h_{m,q}^{-1}(q_j) = m_j.$$

3 Probability

When $\forall A_j \in \mathcal{F}, |A_j| = 1$, then $\mathcal{S}$ is **specific**, and $\Pr(A) := \mathrm{Pl}(A) = \mathrm{Bel}(A)$ is a **probability measure** which is additive in the traditional way

$$\forall A, B \subseteq \Omega, \quad \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Then $\vec{p} = \langle p_i \rangle := \vec{\mathrm{Pl}}$ is a **probability distribution** with additive normalization and operator

$$\sum_i p_i = 1, \qquad \Pr(A) = \sum_{\omega_i \in A} p_i.$$

Theorem 9 If $\mathcal{S}$ is specific, then

$$g_p(A_j) := \omega_i \text{ such that } A_j = \{\omega_i\}$$

is a structural aggregation function.

Theorem 10 If $\mathcal{S}$ is specific, then

$$h_{m,p}(m_j) := m_j = p_j$$

is a numerical aggregation function.

Proposition 11 When p is complete then $\forall \omega_i, \exists A_j = \{\omega_i\}$, and by relabeling, simply

$$g_p^{-1}(\omega_j) = A_j, \qquad h_{m,p}^{-1}(p_j) = m_j.$$

Corollary 12 If p is complete, then $\forall \omega_i, p_i > 0$.

4 Possibility

$\mathcal{S}$ is **consonant** ($\mathcal{F}$ is a **nest**) when (without loss of generality for ordering, and letting $A_0 := \emptyset$) $A_{j-1} \subseteq A_j$. Now $\Pi(A) := \mathrm{Pl}(A)$ is a **possibility measure** and $\eta(A) := \mathrm{Bel}(A)$ is a **necessity measure**. Since results for necessity are dual to those of possibility, only possibility will be discussed in the sequel.

As Pr is additive, so Π is **maximal**:

$$\forall A, B \subseteq \Omega, \quad \Pi(A \cup B) = \Pi(A) \vee \Pi(B),$$

where $\vee$ is the maximum operator. As long as $\mathbf{C}(\mathcal{F}) \neq \emptyset$ (this is required if $\mathcal{F}$ is a nest), then $\vec{\pi} = \langle \pi_i \rangle := \vec{\mathrm{Pl}}$ is a **possibility distribution** with maximal normalization and operator

$$\bigvee_i \pi_i = 1, \qquad \Pi(A) = \bigvee_{\omega_i \in A} \pi_i.$$

Theorem 13 If $\mathcal{S}$ is consonant, then a structural aggregation function $g_\pi \colon \mathcal{F} \mapsto \Omega$ exists.

Theorem 14 If $\mathcal{S}$ is consonant, then

$$h_{m,\pi}(m_j) := \sum_{k=j}^{N} m_k = \pi(\omega_j)$$

is a numerical aggregation function.

Theorem 15 When $\mathcal{S}$ is complete with distribution $q = \pi$, then, using the relabeling convention of (6),

1. $g_\pi(A_j) = A_j - A_{j-1} = \omega_j$,
2. $g_\pi^{-1}(\omega_j) = A_j = \{\omega_1, \omega_2, \ldots, \omega_j\}$,
3. $h_{m,\pi}^{-1}(\pi_j) = m_j = \pi_j - \pi_{j+1}$, where $\pi_{n+1} = 0$ by convention.

Theorem 16 π is complete iff

$$1 = \pi_1 > \pi_2 > \cdots > \pi_n > 0.$$

5 Conclusion

The ideas presented here show the way forward to considering other special cases beyond probability and possibility. In particular, other structural and/or numerical aggregation functions may generate other forms of distributions with their own normalization conditions.

As an example, consider the function

$$g_r(A_j) := A_j \cap A_{j-1} = \omega_j$$

as a structural aggregation function. This specifies "ring-like" random sets where focal elements are linked by sharing exactly one element with each of two other focal elements on "either side" of it. For example, let $\Omega = \{a, b, c, d\}$, and let

$$\mathcal{F} = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}.$$

Since g_r must be one to one, then $\mathcal{F}$ can be constructed if the ordering among the A_j is known. It remains an open question whether a corresponding numerical aggregation function or operator exists in this case.

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