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IN SUPPORT OF AN INDEPENDENT POSSIBILITY THEORY

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ABSTRACT

Possibility theory has traditionally been developed solely in relation to fuzzy theory. This paper will present arguments to support the position that possibility theory should be regarded as related to, but distinct from, both probability and fuzzy theory, all considered as branches of a broader Generalized Information Theory (GIT). On review of Zadeh's original definition of possibility, the role of possibilistic normalization will prove crucial for the articulation of a "regular" view of the relation between fuzzy sets, probability, and possibility valued on $[0,1]$. After consideration of some consequences of and support for this regular view, a series of objections will be addressed, including the weakness of possibilistic normalization, the relation between alpha cuts and random set focal elements, and finally the role of the maximum operator in possibility theory, fuzzy theory, and fuzzy relation composition.

1. Introduction

As is well known, possibility theory was introduced in 1978 in a seminal paper by Zadeh[24]. In it, he first defined not only the possibility distribution, but also most of the issues which have remained central in possibility theory to this day, including possibility measures; joint, marginal, and conditional possibility; and first attempts at a semantics for possibility. Zadeh treated possibility theory strictly in relation to fuzziness, and for him a possibility distribution was an alternative interpretation of a fuzzy set, a "fuzzy restriction" on a variable.

Possibility theory and fuzzy theory grew in close association, with possibility theory regarded generally as a sub-field within fuzzy theory. Sometimes the terms "possibilistic" and "fuzzy" were used synonymously, and beginning in the early 1980's textbooks and anthologies appeared with titles like *Advances in Fuzzy Sets, Possibility Theory, and Applications*[20] and *Fuzzy Set and Possibility Theory: Recent Developments*[23], in which the two mathematical theories were conflated.

There are important historical and methodological reasons why Zadeh's original interpretation of a possibility distribution as a fuzzy set came to dominate possibility theory. This view is still held more or less to this day, and it has certainly resulted in many valuable results and applications. But in the years since Zadeh's paper, alternate mathematical bases for possibility have been developed, including modal logics[16], fuzzy measures[2,3], and consistent or consonant random sets[11]. There are also new competing bases for possibilistic semantics, including modal, natural language, and probabilistic concepts, and objective measurement methods based on consistent set-valued statistics[9].

It is therefore appropriate now, almost twenty years after the founding of possibility theory, to reconsider its place within the broader world of non-probabilistic generalizations of uncertainty and informational concepts—called Generalized Information Theory (GIT) by Klir[14]. This paper will present arguments to support the view of possibility theory as an independent branch of information theory, distinct from, but related to, both probability and fuzziness.

2. Basic Possibilistic Mathematics

It is assumed that the reader is familiar with the basic ideas of fuzzy systems, Dempster-Shafer evidence theory, and mathematical possibility theory[5,15]. Definitions below are to introduce notation only. Further, we generally consider only systems defined on a finite universe of discourse $\Omega = \{\omega_i\}, 1 \le i \le n := |\Omega|$.

The most general mathematical basis for possibility theory currently is fuzzy measure and integral theory, as exemplified by Wang and Klir[21] and Cooman *et al*[2,3]. These two groups generalize the standard possibility theory by extending possibility measures to the positive reals and to lattices respectively. We will not consider these extensions directly here, except to briefly introduce some concepts of fuzzy measures.

A function $\nu: 2^\Omega \mapsto [0,1]$ is a fuzzy measure[21] if $\nu(\emptyset) = 0$ and $\forall A, B \subseteq \Omega, A \subseteq B \rightarrow \nu(A) \le \nu(B)$. Then

$$q_\nu(\omega) := \nu(\{\omega\}) \qquad (1)$$

is a distribution[11] of ν if there exists a distribution operator function $\oplus: [0,1]^2 \mapsto [0,1]$ where $\langle [0,1], \oplus, 0\rangle$ is an Abelian monoid ($\oplus$ is a commutative, associative, operator with identity 0) and, in operator notation,

$$\forall A \subseteq \Omega, \quad \nu(A) = \bigoplus_{\omega \in A} q_\nu(\omega). \qquad (2)$$

When $\oplus$ exists as in Eq. (2), then Eqs. (1,2) establish a bijection between ν and q_ν.

The central tenet of possibility theory is the introduction of a fuzzy measure Π called a possibility measure with a possibility distribution $\pi = q_\Pi$, where $\forall A, B \subseteq \Omega, \Pi(A \cup B) = \Pi(A) \vee \Pi(B)$, and $\oplus = \vee$ is the maximum operator.

One of the richest domains in GIT is that of random set theory (equivalently, Dempster-Shafer evidence theory)[11]. A function $m: 2^\Omega \mapsto [0,1]$ is an evidence function (basic probability assignment) when $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$. Denote a random set generated from an evidence function as $\mathcal{S} := \{\langle A_j, m_j\rangle : m_j > 0\}$, where $\langle \cdot \rangle$ is a vector, $A_j \subseteq \Omega, m_j := m(A_j)$, and $1 \le j \le N := |\mathcal{S}| \le 2^n - 1$. Denote the focal set of $\mathcal{S}$ as $\mathcal{F} := \{A_j : m_j > 0\}$ with focal elements A_j. When $N = n$ then the focal elements A_j can also be indexed by i. The fuzzy measure $\text{Pl}(A) := \sum_{A_j \cap A \neq \emptyset} m_j$, is called a plausibility measure with plausibility assignment (one-point coverage function) $\vec{\text{Pl}} = \langle \text{Pl}_i \rangle := \langle \text{Pl}(\{\omega_i\})\rangle$, where $\text{Pl}_i := \sum_{A_j \ni \omega_i} m_j$.

A fuzzy set denoted $\tilde{F} \tilde{\subseteq} \Omega$ is defined mathematically by the existence of a membership function $\mu_{\tilde{F}}: \Omega \mapsto [0,1]$. We introduce Zadeh's 1978[24] usage here as follows. Given a fuzzy set $\tilde{F} \tilde{\subseteq} \Omega$, then the possibility distribution $\pi_{\tilde{F}}$ based on $\tilde{F}$ is

$$\forall \omega_i \in \Omega, \quad \pi_{\tilde{F}}(\omega_i) := \mu_{\tilde{F}}(\omega_i). \qquad (3)$$

Zadeh defined a possibility measure, denoted here as $\Pi_{\tilde{F}}$, in terms of the possibility distribution, and based on the inverse of the bijective fuzzy measure operator relation from Eq. (2)

$$\forall A \subseteq \Omega, \quad \Pi_{\tilde{F}}(A) := \bigvee_{\omega_i \in A} \pi_{\tilde{F}}(\omega_i) = \bigvee_{\omega_i \in A} \mu_{\tilde{F}}(\omega_i). \tag{4}$$

3. Possibility and Probability

Clearly Zadeh's approach differs from the fuzzy measure approach in that it begins with a possibility distribution, defined as a fuzzy set, and then derives the possibility measure from it. And on the surface this is not a problem, since as we have noted Eqs. (1,2) establish a bijection for $\oplus = \vee$.

Rather, it is when possibility theory is considered in relation to other theories of uncertainty that more questions arise. In particular, there is a desire to develop possibility theory in relation not only to fuzzy theory, but also to probability theory. In many ways, possibilistic formalisms are the strict analogs of stochastic formalisms, where additivity is replaced by "maxitivity".

To make this more explicit, again assuming a general fuzzy set $\tilde{F} \tilde{\subseteq} \Omega$, define

$$\forall \omega_i \in \Omega, \quad r_{\tilde{F}}(\omega_i) := \mu_{\tilde{F}}(\omega_i) \tag{5}$$

as a distribution and an additive fuzzy measure based on $r_{\tilde{F}}$

$$\forall A \subseteq \Omega, \qquad R_{\tilde{F}}(A) := \sum_{\omega_i \in A} r_{\tilde{F}}(\omega_i). \tag{6}$$

It is clear that a similar bijective mapping between the measure and distribution exists in this case, with $\oplus = +$ (proof is omitted for space reasons, but is very simple).

Proposition 7 Assume $\mu_{\tilde{F}}, r_{\tilde{F}}$ and $R_{\tilde{F}}$ as in Eqs. (5,6). Then

$$\forall A, B \subseteq \Omega, \quad R(A \cup B) = R(A) + R(B) - R(A \cap B), \qquad r_{\tilde{F}} = q_{R_{\tilde{F}}}, \tag{8}$$

Clearly, strictly analogously to the possibilistic case, $R_{\tilde{F}}$ and $r_{\tilde{F}}$ demand interpretation as a probability measure and distribution respectively. This result, somewhat trivial in its own right, points the way towards placing possibility and probability theory in the proper context within an overall GIT.

The most important observation is, in fact, the simplest: it is only when fuzzy sets are combined with an appropriate mathematical operator, $\vee$ in the possibilistic case, or $+$ in the stochastic case, that they acquire any significance in terms of fuzzy measures. Presumably a fuzzy set $\tilde{F}$ may be able to be combined with a completely different kind of distribution operator $\oplus$ to yield a different kind of measure.

Therefore the effect of Zadeh's definition is to establish a *synonymous* relation between fuzzy sets and possibility distributions: "possibility distribution" is just another name for "fuzzy set". Other synonyms can also be established, for example "probability distribution" could be just another name for both "fuzzy set" and "possibility distribution", according to the "revised Zadeh definition" of Eq. (5).

This is, in itself, not necessarily a *problem*, but it is clearly *unsatisfactory*. The purpose of introducing definitions is to parsimoniously capture appropriate and meaningful distinctions. Defining possibility distributions *strictly* as fuzzy sets does not do so, since they are not thereby distinguished from other classes of distributions.

4. Normalization

Another observation is that there *is* a problem if we wish to consider $R_{\tilde{F}}$ as a probability measure Pr. While $R_{\tilde{F}}$ is additive, it is not *normal*: $\Pr(\Omega) = 1$ is also required of probability measures, and this forces $\sum_{i=1}^{n} p(\omega_i) = 1$ for the probability distribution p. Letting $p := r_{\tilde{F}}, \Pr := R_{\tilde{F}}$ results in a probability distribution which is almost always either subnormal ($\sum p_i < 1$) or supernormal ($\sum p_i > 1$). So clearly if we wish to use "probability" in this context in any meaningful sense, an additional condition $\sum_{i=1}^{n} \mu_{\tilde{F}}(\omega_i) = 1$ must also hold before we can adopt Eq. (5).

So carrying out a similar reasoning in the possibilistic case, possibilistic normalization $\Pi(\Omega) = 1$ in turn implies $\bigvee_{i=1}^{n} \pi_i = 1$. It follows that Zadeh's definition of possibility in Eq. (3) should be modified to identify possibility distributions as those fuzzy sets which are maxitively, that is, possibilistically, normal: $\bigvee_{i=1}^{n} \mu_{\tilde{F}}(\omega_i) = 1$.

To state the whole argument, we first assume that it is desirable to develop possibility theory in the context of both fuzzy sets and probability, appropriately finding the distinctions and relations among them. Then if subnormal possibility is allowed, the identification of possibility distributions with fuzzy sets is not problematic. But then probability distributions, and other hypothetical distributions of other hypothetical unnormalized fuzzy measures ($\nu(\Omega) \not\equiv 1$), might as well be identified with fuzzy sets, and thus the Zadeh definition captures no significant distinctions. But on the other hand, if possibilistic normalization *is* required, then clearly possibility distributions are identified as maxitively normal fuzzy sets, and probability distributions as additively normal fuzzy sets, leaving the (very large) class of fuzzy sets which are not normal for any distribution operator $\oplus$ as fuzzy sets "proper".

Now most possibility theorists consider normalization a secondary characteristic of possibility distributions, whereas this is decidedly not the case for probability theory. And although there are a great many contexts where normalized fuzzy measures ($\nu(\Omega) \equiv 1$) are produced (for example in random set theory), normalization is not a *logical* requirement for fuzzy measures in general.

But there are at least three serious problems with failing to require normalization of possibility distributions and measures which are valued on $[0, 1]$.

- In process theory (see Sec. 7.6), normalization is an important precondition for satisfaction of basic theorems[10]. Without normalization, possibilistic processes, for example, cannot be defined.
- Semantically, normalization in possibility theory, similar to normalization in probability theory, expresses the condition that all the information available has been accounted for.
- Finally, there *is* a desire among those who are developing possibility theory in the context of possibilistic *information* theory[7] to *retain* that crucial link between possibility and probability, to develop possibilistic information theory analogously to, but still distinct from, traditional (probabilistic) information theory. This strict analogy cannot hold without possibilistic normalization.

5. The Regular View of Fuzziness, Possibility, and Probability

I will describe, for want of a better term, as "regular" the view of the relation between fuzziness, probability, and possibility briefly stated as follows. Any given fuzzy set $\tilde{F}$ *could* define either a probability distribution *or* a possibility distribution, or even *both*, depending on the *properties* of $\mu_{\tilde{F}}$. But in no way does it follow that fuzzy sets are *particular* to possibility theory. On the contrary, it must be recognized that both probability and possibility distributions are *special cases* of fuzzy sets.

Adopting this view requires or leads to the following positions:

- Fuzzy measures, their distributions, and membership functions take values on $[0, 1]$;
- Normalization is necessary for possibility measures and distributions;
- Possibility theory should be developed in a manner analogous to that of probability theory;
- Probability and possibility are special cases of fuzziness;
- And finally, **neither probability nor possibility has a privileged position with respect to fuzziness.**

A major step down this path was expressed (perhaps first) by Kosko[17], who noticed that, obviously enough, *probability distributions are fuzzy sets.* Viewing the power set of Ω as the Boolean lattice of order n, and the "fuzzy power set" (set of all fuzzy subsets or Ω) as the Boolean hypercube of order n, then probability and (normal) possibility distributions are respectively the $(n-1)$-dimensional hyper-tetrahedron with vertices at the Boolean points $\langle 0, 0, \ldots, 1, \ldots, 0, 0\rangle$, and the outermost hypersurfaces of the hypercube which do not intersect the origin.

6. Consequences of and Support for the Regular View

Adoption of the regular view has consequences for the way we view the proper relation among fuzzy, stochastic, and possibilistic systems theories as fields of study. First, it supports the fuzzy community in their long struggle with some members of the traditional information theory community (Cheesman[1], for example) not only for the *significance* of fuzzy theory, but also its *independence* from probability theory. While it would certainly be a retreat on their part to even admit that fuzziness was necessary, albeit only another method in the broader GIT, it would be an outright defeat for them to admit that probability *itself* can be considered as a *case* of fuzziness.

Second, the regular view similarly argues for the independence of possibility theory from *both* fuzzy theory and probability theory, and *against* the traditional view of fuzziness and possibility existing in a special relation.

But in addition, this view *preserves* fuzzy theory as the most general of the three, and therefore the most logically fundamental. Not only is possibility theory preserved as a *special case* of fuzzy systems, but furthermore *all* of *probability* theory can also be claimed by fuzzy systems. Alternatively, probability and possibility can be viewed as those important special classes of fuzzy systems, exactly because they have the properties of (additive or maxitive) normalization.

It is instructive to note, for example, that many applications of fuzzy sets stress the importance of maxitive normalization, of fuzzy sets with a nonempty core. "Fuzzy arithmetic"[13] is built from fuzzy intervals and numbers, which are possibilistically normal. To the extent that normalization is required, then indeed these fuzzy numbers yield to a possibilistic interpretation. But under the regular interpretation, since normalization is a requirement of possibility theory, then by parsimony it is more appropriate to regard fuzzy mathematics as a branch of *possibility* theory proper, rather than as an *application* of *fuzzy* theory.

7. Objections to the Regular View

There are also reasons why possibility and fuzzy sets have been traditionally linked in GIT, and conversely why probability and fuzzy sets have been divorced. These require careful consideration, and lead us to a much deeper understanding of some important issues in fuzzy systems theory.

7.1. Historical Linkage

First and most obviously, probability theory has been in existence for centuries, whereas both fuzzy sets and possibility theory appeared with the development of GIT. As both fuzzy sets and possibility theory are departures from the classical information theory, there is a desire to both group them together, and also distinguish them both from probability. Indeed, much confusion has resulting from the misinterpretation of fuzzy membership grades as probability values, and a great deal of pain is taken by GIT researchers to distinguish them. It is interesting that a corresponding confusion of membership grades with *possibility* values has not troubled these researchers.

7.2. Weakness of Possibilistic Normalization

Possibilistic (maxitive) normalization is weaker than probabilistic (additive) normalization. The measure of the number of possibility distributions on the unit hypercube of dimension n (the hyper-area of the hyper-surfaces occupied by the possibility distributions) is n, while the measure of the number of probability distributions is the hyper-area of the hyper-tetrahedron with side length $\sqrt{2}$ and dimension $n-1$, which is less than n for $n \geq 2$ (for $n = 1$ then $\langle 1\rangle$ is both the only possibility distribution and the only probability distribution).

The proposed possibilistic normalization methods are also much weaker and less distorting than stochastic normalization. In the focused consistent transformation method[8,19] the possibility of a single element of a subnormal possibility distribution is changed to 1, leaving the others unchanged. In the dimensional extension method[11] a unitary value is appended, thus increasing the size of the universe by 1. Geometrically, dimensional extension projects a subnormal fuzzy set to unity in a direction orthogonal to all existing dimensions, while the focused consistent transformations project it to unity on one of the existing dimensions.

An example is shown in Fig. 1 for the subnormal plausibility assignment $\vec{\mathrm{Pl}} = \langle .6, .8\rangle$ regarded as a fuzzy set in the fuzzy power set $[0,1]^{\{x,y\}}$ of $\Omega = \{x, y\}$. There are two focused consistent transformations $\vec{\pi}^x = \langle 1, .8\rangle$ and $\vec{\pi}^y = \langle .6, 1\rangle$, while the dimensional extension is $\vec{\pi}^{n+1} = \langle .6, .8, 1\rangle$ for $z = \omega_3$.

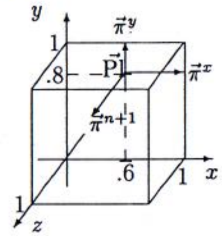

Figure 1: Dimensional extension and focused consistent transformation normalization.

Possibilistic normalization is so weak that it can be easily accommodated or overlooked, supporting the interpretation of general fuzzy sets as possibility distributions. But it is our contention that for measures and distributions on $[0,1]$ it can no more be ignored as a requirement in possibility theory than stochastic normalization can in probability theory. And while it is true that there are far more possibility distributions in the fuzzy power set than probability distributions, nevertheless it is also true that there are far more fuzzy sets which are neither possibility nor probability distributions, and which fill the n-hypercube.

7.3. Fuzzy Set Normalization

It is also suggestive that the definition of "normalization" for a fuzzy set $\tilde{F}$ is identical to that in possibility theory: $\bigvee_{i=1}^{n} \mu_{\tilde{F}}(\omega_i) = 1$. Thus the criteria for $\tilde{F}$ to be fuzzy-set-normal is both necessary and sufficient for the Zadeh-possibility distribution $\pi_{\tilde{F}}$ to be possibilistically normal. This seems on the surface to be an historical accident in the usage of the term "normal" in each of these mathematical contexts.

But the ease of maxitive normalization is reinforced by the common treatment of fuzzy sets on a continuous universe, for example with the ubiquitous fuzzy numbers. The height of the fuzzy set $\sup_{\omega \in \Omega} \mu_{\tilde{F}}(\omega)$ is the most obvious feature of the curve, and the height being maximal is equivalent to possibilistic normalization. In the same situation stochastic normalization results in regarding $\mu_{\tilde{F}}$ as a probability density, so that for stochastic normalization $\int_\Omega \mu_{\tilde{F}}(\omega)\, d\omega = 1$ would have to be satisfied. A unitary area is not an obvious, visual feature of a curve.

7.4. Alpha Cuts and Focal Elements

One of the most powerful arguments *for* a special relation between possibility and fuzzy sets is the fact that the alpha cuts of a fuzzy set $\tilde{F}$

$$\tilde{F}_\alpha := \{\omega_i : \mu_{\tilde{F}}(\omega_i) \geq \alpha\} \subseteq \Omega \tag{9}$$

form a nest, in that $\alpha_1 > \alpha_2 \rightarrow \tilde{F}_{\alpha_1} \subseteq \tilde{F}_{\alpha_2}$. Thus if $\tilde{F}$ is fuzzy-set-normal and ordered (without loss of generality) so that $1 = \mu_{\tilde{F}}(\omega_1) \geq \mu_{\tilde{F}}(\omega_2) \geq \cdots \geq \mu_{\tilde{F}}(\omega_n)$, then the alpha cuts $A_i := \tilde{F}_\alpha = \{\omega_1, \omega_2, \ldots, \omega_i\}$ are the focal elements of a consonant random set $\mathcal{S}$ with evidence function $m(A_i) = \mu_{\tilde{F}}(\omega_i) - \mu_{\tilde{F}}(\omega_{i+1})$ ($\mu_{\tilde{F}}(\omega_{n+1}) := 0$ by convention), which in turn generates a possibility measure $\Pi = \mathrm{Pl}$ with distribution (one-point coverage function of $\mathcal{S}$) $\pi = \mu_{\tilde{F}} = \pi_{\tilde{F}}$. This result has been used to justify a special equivalence between a fuzzy set and a consonant random set, and thus the corresponding distribution.

But there are problems with this view. In particular, the alpha cuts of a *subnormal* fuzzy set *also* form a nest, but this does not mean that a consonant random set can be constructed from it.

Theorem 10 Assume $\tilde{F}$ with $\mu_{\tilde{F}}(\omega_1) \geq \mu_{\tilde{F}}(\omega_2) \geq \cdots \geq \mu_{\tilde{F}}(\omega_n) \geq \mu_{\tilde{F}}(\omega_{n+1}) := 0$, and let $A_i := \{\omega_1, \omega_2, \ldots, \omega_i\}, m(A_i) := \mu(\omega_i) - \mu(\omega_{i+1})$. Then $\{\langle A_i, m(A_i)\rangle\}$ is a random set iff $\tilde{F}$ is fuzzy-set-normal.

Proof: In general, $\sum_i m(A_i) = \sum_i (\mu_{\tilde{F}}(\omega_i) - \mu_{\tilde{F}}(\omega_{i+1})) = \mu_{\tilde{F}}(\omega_1)$. So if $\tilde{F}$ is normal, then $\sum_i m(A_i) = 1$ and $\forall i, A_i \neq \emptyset$, so $\{\langle A_i, m(A_i)\rangle\}$ is a random set. If $\tilde{F}$ is subnormal, then $\sum_i m(A_i) = \mu_{\tilde{F}}(\omega_1) < 1$, which violates the definition of a random set. ■

So the fact that the alpha cuts of a fuzzy set—*even a probability distribution*—form a nest in no way lessens the normalization requirement for possibility. The nesting of the alpha cuts is an *artifact* of the way they are constructed, and in particular on the reliance of the $\geq$ operator in Eq. (9). Wierman[22] has recently suggested that alpha cuts can be generalized to other relational operators other than $\geq$.

Finally, while it is true that the alpha cuts of a fuzzy set can be mapped to the focal elements of a consonant random set as above, there are *other* kinds of random sets which can also be constructed from $\mu_{\tilde{F}}$. In particular, the sets $A_i = \{\omega_i\}$ can be focal elements of a *specific* random set with evidence function $m(A_i) = \mu_{\tilde{F}}(\omega_i)$, which in turn generates a probability measure $\Pr = \mathrm{Pl}$ with distribution (one-point coverage function of $\mathcal{S}$) $p = \mu_{\tilde{F}} = r_{\tilde{F}}$. Strictly analogously to the possibilistic case, *stochastic* (additive) normalization must hold here.

Theorem 11 Assume $\tilde{F}$ with $A_i := \{\omega_i\}, m(A_i) := \mu_{\tilde{F}}(\omega_i)$. Then $\{\langle A_i, m(A_i)\rangle\}$ is a random set iff $\sum_{i=1}^{n} \mu_{\tilde{F}}(\omega_i) = 1$.

Proof: In general, $\sum_i m(A_i) = \sum_i \mu_{\tilde{F}}(\omega_i)$, so if $\sum_{i=1}^{n} \mu_{\tilde{F}}(\omega_i) = 1$, then because $\forall i, A_i \neq \emptyset, \{\langle A_i, m(A_i)\rangle\}$ is a random set. But if $\sum_i \mu_{\tilde{F}}(\omega_i) \neq 1$, then $\sum_i m(A_i) \neq 1$, which violates the definition of random set. ■

To summarize, neither probability nor possibility is wedded to fuzzy set theory: each is a *case* of it. A probability distribution yields a consonant focal set based on the ordering of the p_i just as well as a possibility distribution, almost always resulting in a subnormal possibility measure. Similarly, a possibility distribution yields a specific focal set as easily as a probability distribution, almost always resulting in a *supernormal* probability distribution.

7.5. The Possibilistic Operator and Fuzzy Unions

The most obvious reason to conflate fuzzy theory with possibility theory is the fact that $\vee$, as the possibilistic operator, is also the canonical fuzzy set union operator $\mu_{\tilde{F}\cup\tilde{G}} = \mu_{\tilde{F}} \vee \mu_{\tilde{G}}$. But in general, the fuzzy union operator can be *any* t-conorm $\sqcup$, not just $\vee$. It is true that $\vee$ is canonical, and in some ways more justified than other conorms, but *formally* any will suffice, and in practice others are used.

This contrasts with both possibility theory and fuzzy measure theory. $\vee$ is the *unique* possibilistic operator; indeed, it *defines* the very domain of applicability of possibility theory. And while $\vee$ is indeed a conorm, in the general theory of distributions of fuzzy measures, $\oplus$ in Eq. (2) need *not* be a conorm. Indeed, the other fundamental special case has $\oplus = +$, and while the bounded sum $(x+y) \wedge 1$ is a conorm, $+$ in general is *not*.

7.6. The Possibilistic Operator and Fuzzy Matrix Composition

In fuzzy relation composition, given two (here square) fuzzy relations $\tilde{R}, \tilde{S} \tilde{\subseteq} \Omega^2$, then their composition $\tilde{T} = \tilde{R} \circ \tilde{S}$ is given by

$$\mu_{\tilde{T}}(\omega_i, \omega_j) = \bigvee_{k=1}^{n} \mu_{\tilde{R}}(\omega_i, \omega_k) \sqcap \mu_{\tilde{S}}(\omega_k, \omega_j), \tag{12}$$

where $\sqcap$ is a t-norm, usually $\wedge$. In fuzzy processes, as used in fuzzy automata[4], a fuzzy state vector $\vec{\tilde{\phi}} \tilde{\subseteq} \Omega$ is composed with an $n \times n$ fuzzy transition matrix $\tilde{\sigma} \tilde{\subseteq} \Omega^2$ to generate a next fuzzy state vector $\vec{\tilde{\phi}}' = \vec{\tilde{\phi}} \circ \tilde{\sigma}$. Possibilistic processes then follow when $\vec{\tilde{\phi}}$ is possibilistically normal, and $\tilde{\sigma}$ is a conditional possibility matrix (the columns of $\tilde{\sigma}$ are possibilistically normal)[11].

So it appears that possibilistic processes are a case of fuzzy processes, and thus possibility theory *does* have a privileged position with respect to fuzzy theory. But we must question whether the use of $\vee$ and $\langle \vee, \sqcap \rangle$ operations in fuzzy relation composition follows the use of $\vee$, and particularly $\langle \vee, \wedge \rangle$, in fuzzy sets and logic. In fuzzy sets and logic the negation operator $\neg$, with usually $\neg x = 1 - x (x \in [0,1])$, plays a crucial role. Then, given a conjunction $\sqcap$ (respectively disjunction $\sqcup$), the DeMorgan property $\neg(x \sqcap y) = \neg x \sqcup \neg y (x, y \in [0,1])$ is used to derive the dual disjunction $\sqcup$ (respectively conjunction $\sqcap$). It is, in fact, systems of the form $\langle \sqcup, \sqcap, \neg \rangle$ (usually $\langle \vee, \wedge, 1 - \cdot \rangle$), with $\sqcup$ and $\sqcap$ DeMorgan dual in $\neg$, which form the basis of fuzzy theory.

But $\vee$ and a general norm $\sqcap$ are not generally DeMorgan. Furthermore, whereas in fuzzy systems any conorm $\sqcup$ can be used for disjunction, in relation composition *only* the $\vee$ operator is used for generalized addition, and there is *no* use of negation.

What must be recognized is that fuzzy relation composition generalizes mathematically in abstract algebra to semiring structures, as used, for example, in generalized process theory[11] and automata theory[18]. A semiring on $[0,1]$ is a pair of associative operators $\langle \oplus, \otimes \rangle$ where $\oplus$ is commutative with identity 0 and $\otimes$ distributes over $\oplus$. Clearly fuzzy relation composition follows from $\langle \oplus, \otimes \rangle = \langle \vee, \sqcap \rangle$, since $\vee$ is distributed over by all norms $\sqcap$. But there are other semiring structures, including the additive semiring $\langle +, \times \rangle$ of stochastic process theory, which do not rely on $\vee$, or norms and conorms in general.

Historically, fuzzy relation composition developed using $\langle \vee, \wedge \rangle$ composition, and was only later generalized to $\langle \vee, \sqcap \rangle$. So the identification of possibility with general fuzzy concepts is understandable. But the essential difference between process theory and set theory and logic is that the former relies essentially on distributivity, and not at all on DeMorgan (or negation), while the latter relies essentially on DeMorgan (and also negation), but not at all on distributivity.

In fact, $\langle \vee, \wedge \rangle$ is the *only* $\langle \sqcup, \sqcap \rangle$ pair which is *both* distributive *and* DeMorgan. The point is that $\vee$ is a very weak and flexible operator. It can serve in many algebraic capacities, as a distribution operator or a conorm. But this fact alone is not *prima facia* evidence to support the identification of possibility theory with fuzzy theory.

Our argument is not that failing to adopt the regular view limits *possibility theory*, but rather it limits *fuzzy theory*: as with the identification of distributions as fuzzy sets, fuzzy theory is *more general*. Whereas possibility theory is wedded to $\vee$, fuzzy theory in general is *not*, and should not be.

Instead, from a "pure" GIT perspective, we should not call $\langle \vee, \sqcap \rangle$ processes "fuzzy", but rather **general possibilistic**, to distinguish them from proper (normalized) possibilistic processes. In turn, the term "fuzzy process" should be elevated to include *all* of the processes which take values on $[0,1]$, and use a general fuzzy relation $\tilde{\sigma}$ to operate on a fuzzy set $\vec{\tilde{\phi}}$ and create a new fuzzy set $\vec{\tilde{\phi}}'$, be it a possibility distribution, probability distribution, or general fuzzy set. As with Kosko's realization that probability distributions are fuzzy sets, a central point here is that *additive* processes should be regarded as a branch of *fuzzy* systems theory!

7.7. Fuzzy Subset Intersection

Another argument used to support a special relation between fuzziness and possibility is as follows[5]. Given a crisp subset $F \subseteq \Omega$, then $\forall A \subseteq \Omega$,

$$\Pi_F(A) := \begin{cases} 1, & A \cap F \neq \emptyset \\ 0, & A \cap F = \emptyset \end{cases} \tag{13}$$

is a possibility measure. Since

$$\Pi_F(A) = \bigvee_{\omega \in \Omega} \chi_F(\omega) \sqcap \chi_A(\omega), \tag{14}$$

where χ is the crisp characteristic function, therefore we should consider generalizing

F to a fuzzy subset $\tilde{F} \tilde{\subseteq} \Omega$, whereupon we observe

$$\Pi_{\tilde{F}}(A) := \bigvee_{\omega \in \Omega} \mu_{\tilde{F}}(\omega) \sqcap \chi_A(\omega) = \bigvee_{\omega \in A} \mu_{\tilde{F}}(\omega) \tag{15}$$

is also a possibility measure.

It is true that the characteristic functions of non-empty crisp subsets are crisp ($\pi \in \{0,1\}$) possibility distributions. But this does not support the view that possibility is special for fuzzy sets. It is not the introduction of a fuzzy set $\tilde{F}$ in Eq. (15) which results in a possibility measure; that is present in Eq. (14) when dealing strictly with crisp subsets. So it should not be surprising that $\vee$ and possibility distributions are retained on generalization to Eq. (15).

The observation about the possibilistic nature of non-empty crisp subsets is reflected in the fact that nondeterministic processes and interval analysis are crisp specializations of possibilistic processes and fuzzy intervals (themselves possibilistic, as discussed above) respectively[12]. Indeed, these two fields should be added to the (apparently ever growing) domain of possibility theory.

The argument here is essentially equivalent to that from the previous section on process theory and that leading to Eq. (7). If we generalize Eq. (15) again for a fuzzy subset $\tilde{A} \tilde{\subseteq} \Omega$, then $\langle \vee, \sqcap \rangle$ should be generalized to $\langle \oplus, \otimes \rangle$ semiring composition

$$\nu_{\tilde{F}}(\tilde{A}) := \bigoplus_{\omega \in \Omega} \mu_{\tilde{F}}(\omega) \otimes \mu_{\tilde{A}}(\omega). \tag{16}$$

Indeed, one could define an *additive* operation on a fuzzy set analogously to Eq. (15)

$$R_{\tilde{F}}(A) := \sum_{\omega \in \Omega} \mu_{\tilde{F}}(\omega_i) \cdot \chi_A(\omega) = \sum_{\omega \in A} \mu_{\tilde{F}}(\omega_i) \tag{17}$$

which is by Eq. (7) a (non-normal) *probability* measure.

It is still the case that fuzzy theory, properly conceived, essentially involves functions taking values in $[0,1]$, *but not necessarily the maximum operator*. So all of this section is discussing this broad fuzzy theory, with its stochastic ($\oplus = +$) component and its possibilistic ($\oplus = \vee$) component, itself including the further crisp special cases of subsets, intervals, and nondeterministic processes[11].

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9. References

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LOGICS WITH FUZZY MODALITIES

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ABSTRACT

The article is devoted to constructing of the propositional axiomatic system K_Ω with fuzzy modalities over arbitrary complete Heyting algebra Ω. This axiomatic system consists of formulas of two kinds: ordinary fuzzy modal formulas and evaluated fuzzy modal formulas. Ordinary fuzzy modal formulas of this system are constructed over countable set of propositional variables, using Boolean connectives (negation, conjunction, disjunction, implication and equivalence) and unary modal connectives, which are indexed by elements of some Heyting algebra Ω: for every $a \in \Omega$ there exists an unary modal connective $\Box_a$. An evaluated fuzzy modal formula is a pair (A, a), where A is arbitrary ordinary fuzzy modal formula, and a is an arbitrary element of the Heyting algebra Ω.
It is introduced a fuzzy semantics for this propositional axiomatic system with fuzzy modalities, namely – fuzzy Kripke models. It is introduced the concept of measure of truth of a formula of the language of fuzzy modal logic (FML) in a fuzzy Kripke model.
The main result of the paper is the following: the set of all evaluated fuzzy modal formulas that are true in every fuzzy Kripke model, is equal to the logic K_Ω.

1. Introduction

Using of tools of modal logic for solving the problem of representation of fuzzy and uncertain information and for approximate reasoning were considered in various papers (see, for example, [1–12]).

In the present article is delivered another approach to this problem. The article is devoted to constructing of the propositional axiomatic system K_Ω with fuzzy modalities over arbitrary complete Heyting algebra Ω and to proving of a completeness theorem for this system. This axiomatic system consists of formulas of two kinds: ordinary fuzzy modal formulas and evaluated fuzzy modal formulas. Ordinary fuzzy modal formulas of this system are constructed over countable set of propositional variables, using Boolean connectives (negation, conjunction, disjunction, implication and equivalence) and unary modal connectives, which are indexed by elements of some Heyting algebra Ω: for every $a \in \Omega$ there exists an unary modal connective $\Box_a$. For every formula A of our language the formula $\Box_a A$ is read as "measure of necessity of the formula A is equal to a", or "A is necessary up to a". For every $a \in \Omega$ there can be introduced the dual fuzzy modal connective $\Diamond_a$, and for every formula A of our language the formula $\Diamond_a A$ can be read as "measure of possibility of the formula A is equal to a", or "A is possible up to a". But in the present article we shall consider