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Possibilistic Normalization of Inconsistent Random Intervals†

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1. Introduction

Random set theory (and the mathematically isomorphic Dempster-Shafer evidence theory) is one of the most satifsactory mathematical grounds for fuzzy measure theory in general, and possibility theory in particular. Random sets, seen as set-valued random variables, generate belief and plausibility fuzzy measures in general, probability measures when specific, and possibility measures when consonant. Similarly, the one-point traces of random sets are fuzzy sets which are probability distributions when specific and possibility distributions when consonant.

But the traces of consistent random sets (those with non-empty global intersections) are also possibility distributions, and they yield unique canonical possibility measures which approximate their plausibility measures. Inconsistent random sets yield traces which are possibilistically sub-normal, and require possibilistic normalization.

In this paper we first introduce two new normalization methods for finite random sets. Focused consistent transformations, which have the effect of elevating a particular normalizing element to unity, have been considered previously as possibilistic normalization methods for finite random sets [1]. Selecting the universe element with maximum plausibility as the focus is justified by its order preserving property. Then, dimensional extension appends the normalizing element, embedding the original random set in a higher dimensional space.

We next extend focused consistent transformation normalization to random half-open interval subsets of $\mathbb{R}$ with finite base spaces. Traces of consistent random intervals are possibilistic histograms [3]. Although $\mathbb{R}$ is uncountable, a random interval is a finite collection of interval subsets. A set of intervals which partition the support of the random interval is determined by taking all intersections of the focal elements recursively [3], and these intervals form the atoms of a lattice whose operations are interval intersection and concatenation (over gaps). In this space focused consistent tranformations are modified to require an atomic normalizing interval, and set union is replaced with concatenation.

2. Mathematical Preliminaries

2.1. Random Sets

Assume a finite universe of discourse $\Omega := \{\omega_i\}, 1 \leq i \leq n := |\Omega|$ and evidence function $m\colon 2^\Omega \mapsto [0,1]$ where $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$. Then $\mathcal{S} := \{\langle A_j, m_j \rangle\}$ is a random set where $1 \leq j \leq N := |\mathcal{S}| \leq 2^n$, $m(A_j) > 0$, and $m_j := m(A_j)$. The focal set is $\mathcal{F}(\mathcal{S}) := \{A_j\}$ with core $\mathbf{C}(\mathcal{F}(\mathcal{S})) := \bigcap_{A_j \in \mathcal{F}(\mathcal{S})} A_j$ and support $\mathbf{U}(\mathcal{F}(\mathcal{S})) := \bigcup_{A_j \in \mathcal{F}(\mathcal{S})} A_j$. Random sets are mathematically isomorphic to Dempster-Shafer bodies of evidence.

Given a random set $\mathcal{S}$, define the plausibility measure $\mathrm{Pl}(A) := \sum_{A_j \not\perp A} m_j$ where $A \perp B := A \cap B = \emptyset$ and plausibilistic trace $\rho\colon \Omega \mapsto [0,1]$ where $\rho(\omega_i) := \mathrm{Pl}(\{\omega_i\})$, and in vector form $\vec{\rho} = \langle \rho_i \rangle$, with $\rho_i := \rho(\omega_i)$.

As an example, let $\Omega := \{x, y, z\}$ and consider the random set $\mathcal{S} := \{\langle \{x\}, .5 \rangle, \langle \{x, z\}, .3 \rangle, \langle \{y, z\}, .2 \rangle\}$.

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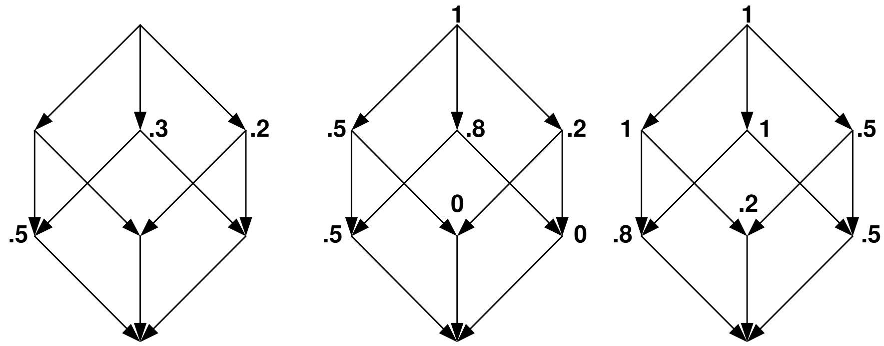

Figure 1. (Left) The example random set. (Center) Bel. (Right) Pl.

This can be depicted graphically as in the left of Fig. 1, where the power set 2^Ω is represented as a Boolean 3-cube. Each node represents a subset $A \subseteq \Omega$ weighted with its value $m(A)$ (values for $m(A) = 0$ are omitted). In the center and right of are the corresponding cubes for Bel and Pl respectively.

When $\mathcal{F}(\mathcal{S})$ is specific, so that $\forall |A_j| = 1$, then Pl is a probability measure with probability distribution $p := \rho$. When $\mathcal{F}(\mathcal{S})$ is a nest, so that the A_j are ordered with $A_{j-1} \subseteq A_j$, where $A_0 := \emptyset$, then $\Pi :=$ Pl is a possibility measure with $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$, and $\pi := \rho$ is its possibility distribution with $\Pi(A) = \bigvee_{\omega_i \in A} \pi(\omega_i)$ and normalization $\bigvee_{i=1}^n \pi(\omega_i) = 1$. But when $\mathcal{F}(\mathcal{S})$ is merely consistent, with $\mathbf{C}(\mathcal{F}(\mathcal{S})) \neq \emptyset$, then $\pi := \rho$ is still a maximum normalized possibility distribution, even though Pl is not necessarily a possibility measure. Then there is a unique canonical possibility measure Π^* derived from π which approximates Pl [1].

Finally, when $\mathcal{F}(\mathcal{S})$ is inconsistent, so that $\mathbf{C}(\mathcal{F}(\mathcal{S})) = \emptyset$, then ρ is possibilistically sub-normal with $\bigvee_{i=1}^n \rho_i < 1$. Then approximations are available which act as possibilistic normalization methods. A minimal possibilistic normalization method is available by introducing a minimal core $\mathbf{C}(\mathcal{S}) = \{\omega_i\}$ for some focus $\omega_i \in \Omega$.

2.2. Consistent Transformations

Given a random set $\mathcal{S}$, a consistent transformation $\mathcal{S} \mapsto \hat{\mathcal{S}}$ creates a new random set $\hat{\mathcal{S}}$, with focal set $\hat{\mathcal{F}} := \mathcal{F}(\hat{\mathcal{S}})$ and evidence function $\hat{m}$, when an evidential claim $\langle A, m(A)\rangle \in \mathcal{S}$ is moved to a new focal element $\hat{A} \in \hat{\mathcal{F}}$, where $\hat{A} \supseteq A$, according to the algorithm:

1 $\hat{m} := m$.
2 $\hat{m}(A) := 0$.
3 $\hat{m}(\hat{A}) := \hat{m}(\hat{A}) + m(A)$.

Given a focus $\omega_k \in \Omega$, a focused consistent transformation creates a new consistent random set $\hat{\mathcal{S}}_k$ with evidence function $\hat{m}^k$ by affecting the transforms $\forall A_j \in \mathcal{F}, A_j \mapsto \hat{A}_j := A_j \cup \{\omega_k\}$. It follows that [1]

$$\forall A \subseteq \Omega, \qquad \hat{m}^k(A) = \begin{cases} m(A) + m(A - \{\omega_k\}), & \omega_k \in A \\ 0, & \omega_k \notin A \end{cases},$$

$$\vec{\rho} = \langle \rho_1, \rho_2, \ldots, \rho_k, \ldots, \rho_n \rangle \quad \mapsto \quad \vec{\pi} = \langle \rho_1, \rho_2, \ldots, 1, \ldots, \rho_n \rangle . \tag{2.1}$$

An example is shown in Fig. 2 for $\Omega = \{x, y, z\}$ and an original random set $\mathcal{S} := \{\langle \{x\}, .1\rangle, \langle \{x, y\}, .7\rangle, \langle \{z\}, .2\rangle\}$ with plausibility assignment $\vec{\rho} = \langle .8, .7, .2\rangle$. The approximations yield respectively

$$\vec{\pi} = \langle 1, .7, .2\rangle, \quad \langle .8, 1, .2\rangle, \quad \langle .8, .7, 1\rangle .$$

2.3. Random Intervals

Let $\mathcal{D} := \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}, a < b\}$ be the class of half-open intervals. Then a random interval $\mathcal{A}$ is a random set on $\Omega = \mathbb{R}$ with $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{D}$, or in other words a random left-closed interval subset of $\mathbb{R}$. Denote the focal sets of $\mathcal{A}$ as $A_j = [l_j, r_j) \subseteq \mathbb{R}, l_j < r_j$.

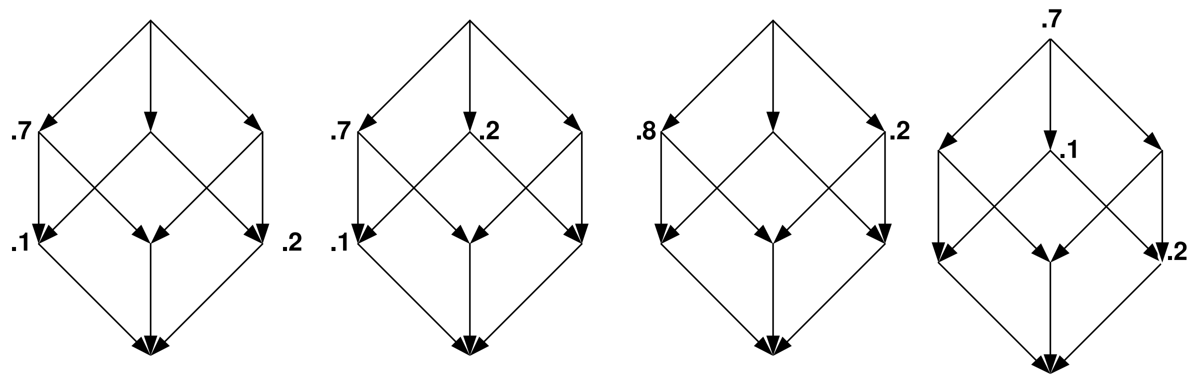

Figure 2. (Left) Original random set $\mathcal{S}$ (Center Left) $\hat{\mathcal{S}}_x$. (Center Right) $\hat{\mathcal{S}}_y$. (Right) $\hat{\mathcal{S}}_z$.

For random intervals, we modify the concept of the plausibility assignment slightly to be the plausibilistic trace $\rho_{\mathcal{A}}\colon \mathbb{R} \mapsto [0,1]$ (or just ρ when clear from context) where

$$\forall x \in \mathbb{R}, \quad \rho_{\mathcal{A}}(x) := \mathrm{Pl}(\{x\}) = \sum_{A_j \ni x} m_j. \tag{2.2}$$

When $\mathcal{A}$ is consistent, then $\pi_{\mathcal{A}} := \rho_{\mathcal{A}}$ is a possibility distribution called a possibilistic histogram, and is a fuzzy interval [3].

3. New Possibilistic Normalization Methods on Finite Random Sets

We first present two new methods for possibilistic normalization of finite random sets.

3.1. Maximum Plausibility Focus Selection

With focused consistent transformations, the general question becomes how to select the focus ω_k. Previously we considered selecting ω_k by the principle of uncertainty invariance, so as to make the total information content of $\hat{\mathcal{S}}$ as close as possible to that of $\mathcal{S}$ [1, 2]. But Ramer and Puflea-Ramer [4] have suggested that it is also reasonable to select as a focus that ω_k with maximal plausibility. This is the focus which distorts the ordering of the original distribution as little as possible.

Corollary 3.3.1 Assume a random set $\mathcal{S}$ with plausibilistic trace $\vec{\rho}$. Derive a possibility distribution by a focused consistent transformation $\vec{\rho} \mapsto \vec{\pi}$ with focus ω_k. Then $\omega_k = \max_{\omega_i \in \Omega} \rho_i$ iff the rank ordering of the ρ_i is preserved in the π_i, so that

$$\rho_{i_1} \geq \rho_{i_2} \rightarrow \pi_{i_1} \geq \pi_{i_2}, \qquad 1 \leq i_1, i_2 \leq n. \tag{3.1}$$

Proof. Denote $\omega_l := \max_{\omega_i \in \Omega} \rho_i$. **Case 1:** Let (3.1) hold. If $\omega_k \neq \omega_l$, then $\rho_k \leq \rho_l$, but $\pi_l \leq \pi_k = 1$, which violates (3.1). Therefore $\omega_k = \omega_l$. **Case 2:** Let $\omega_k = \omega_l$. Obviously $\forall i_1, i_2 \neq l, \pi_{i_1} = \rho_{i_1}$ and $\pi_{i_2} = \rho_{i_2}$, and so (3.1) holds for them. And $\forall i \neq l$, both $\rho_l \geq \rho_i$ and $1 = \pi_l \geq \pi_i$. Therefore (3.1) holds in general. ■

If the example in Fig. 2 with $\vec{\rho} = \langle .8, .7, .2 \rangle$, maximum plausibility would select $\hat{\mathcal{S}}_x$ and $\vec{\pi} = \langle 1, .7, .2 \rangle$.

3.2. Dimensional Extension

A consistent transformation requires the modification of at least one of the ρ_i, which is changed to 1 in order to possibilistically normalize $\vec{\rho}$. However, it is possible to provide a maximum normalized element in a manner which does *not* disrupt the other ρ_i at all, by simply leaving them all unchanged, but instead *adding* a new element $\rho_{n+1} = 1$.

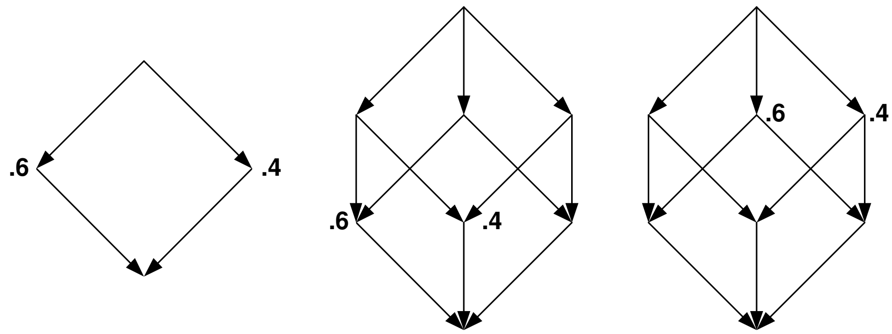

Figure 3. (Left) An inconsistent random set $\mathcal{S}$ in a normal universe Ω. (Center) In an extended universe Ω'. (Right) Its dimensional extension $\hat{\mathcal{S}}_{n+1}$.

Definition 3.3.2 (Dimensional Extension) Given a possibilistically subnormal plausibility distribution $\vec{\rho}$, let $\vec{\pi}' = \langle 1 \rangle + \vec{\rho}$, where $+$ in this context is vector concatenation.

The effect is to replace the universe of discourse Ω with a new universe $\Omega' = \Omega \cup \{\omega_{n+1}\}$, and create a new plausibility assignment which is a possibility distribution $\vec{\rho}' = \langle \rho'_i \rangle$, where $\rho'_1 = 1$ and $\rho'_i = \rho_{i-1}, 2 \le i \le n+1$.

Actually, the correct perspective is not so much that a new element ω_{n+1} is being added, as it is that a random set already defined on Ω', but for which $\exists i, \rho_i = 0$, is consistently transformed with a focus $\omega_k = \omega_{n+1}$, effecting by (2.1) the transformation

$$\vec{\rho} = \langle \rho_1, \rho_2, \ldots, \rho_n, 0 \rangle \mapsto \vec{\pi} = \langle \rho_1, \rho_2, \ldots, \rho_n, 1 \rangle, \tag{3.2}$$

where $\vec{\pi}$ has yet to be appropriately ordered.

Corollary 3.3.3 Given an inconsistent random set $\mathcal{S}$ defined on $\Omega' = \Omega \cup \{\omega_{n+1}\}$ such that

$$\forall A_j \in \mathcal{F}, \quad \omega_{n+1} \notin A_j, \tag{3.3}$$

then the focused consistent transformation $\mathcal{S} \mapsto \hat{\mathcal{S}}_{n+1}$ effects the transform of $\vec{\rho} \mapsto \vec{\pi}$ as in (3.2).

Proof. $\rho_{n+1} = \sum_{A_j \ni \omega_{n+1}} m_j = 0$, so that $\vec{\rho}$ is as in (3.2). The result follows from (2.1), once $\vec{\pi}$ is sorted. ■

From the condition (3.3) above, $\mathcal{S}$, while technically defined on Ω', actually has weight only for $A \subseteq \Omega$, and so exists confined to the simplex $2^{\Omega} \subseteq 2^{\Omega'}$. Dimensional extension ((3.3.2) and (3.3.3)) projects $\mathcal{S}$ into the rest of the space involving the new element ω_{n+1}.

As an example, consider the random set $\mathcal{S} = \{\langle \{x\}, .6 \rangle, \langle \{y\}, .4 \rangle\}$ defined on $\Omega = \{x, y\}$ with $\vec{\rho} = \langle .6, .4 \rangle$. From dimensional extension (3.3.2), $\pi' = \langle 1, .6, .4 \rangle$ (once π' is sorted) defined on $\Omega' = \{x, y, z\}$. The final random set is $\hat{\mathcal{S}}_{n+1} = \{\langle \{x, z\}, .6 \rangle, \langle \{y, z\}, .4 \rangle\}$, as shown in Fig. 3. When $\mathcal{S}$ is taken to be in Ω', then $\vec{\rho} = \langle .6, .4, 0 \rangle$, as shown in the figure.

Geometrically, dimensional extension projects a subnormal fuzzy set to unity in a direction orthogonal to all existing dimensions, while focused consistent transformations projects it to unity on one of the existing dimensions. An example is in Fig. 4 for the subnormal plausibility assignment $\vec{\rho} = \langle .6, .8 \rangle$ regarded as a fuzzy set in the fuzzy power set $[0,1]^{\{x,y\}}$. There are two focused consistent transformations $\vec{\pi}^x = \langle 1, .8 \rangle$ and $\vec{\pi}^y = \langle .6, 1 \rangle$. The dimensional extension is $\vec{\pi}^{n+1} = \langle .6, .8, 1 \rangle$ for $z = \omega_3$.

4. Normalization of Random Intervals

Now consider possibilistic normalization of the trace of an inconsistent random interval. When $\mathcal{F}(\mathcal{A})$ is inconsistent then focused consistent transformations must be modified to take into account the linear order on $\mathbb{R}$.

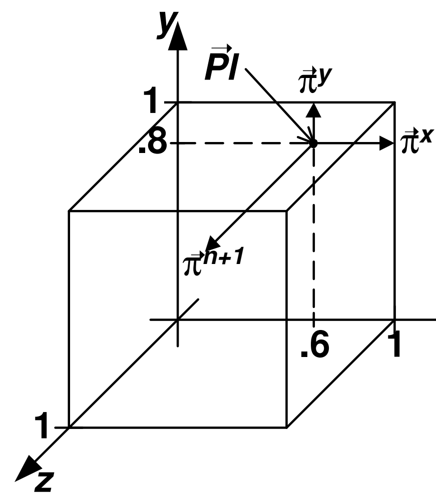

Figure 4. Dimensional extension and focused consistent transformation normalization.

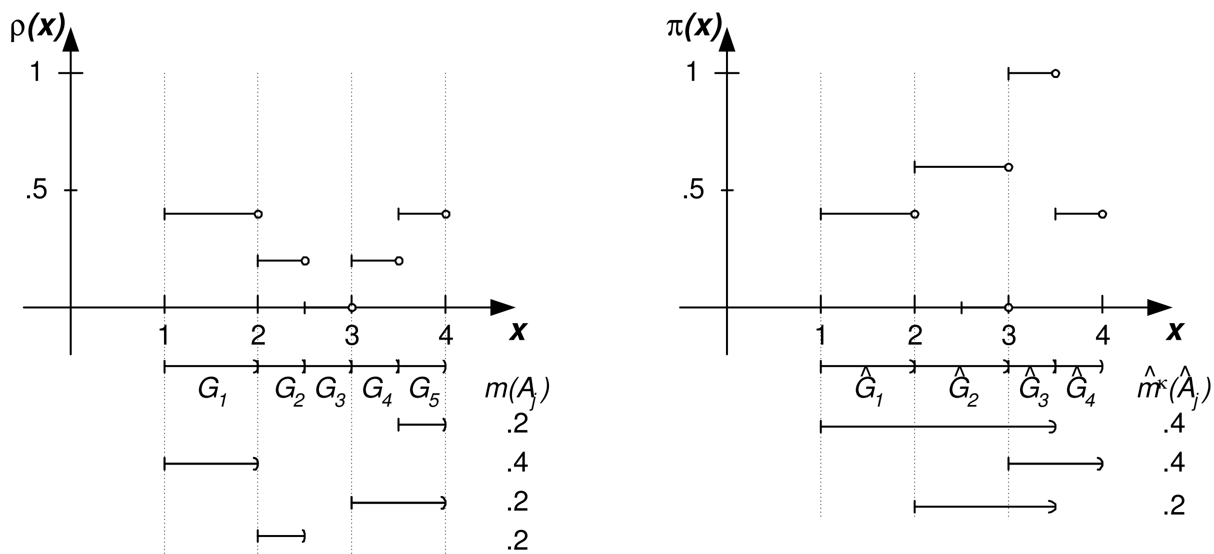

Figure 5. (Left) An inconsistent random interval. (Right) Its focused consistent transformation for $\kappa = 4$.

4.1. The Form of Possibilistic Histograms

Because $\mathcal{F}(\mathcal{A})$ is a finite collection, the trace ρ is fully defined by the finite collection of endpoints $\{l_j, r_j\}$. After global reordering and removal of duplicates, these endpoints determine a class $\Gamma = \{G_k\} \subseteq \mathcal{D}$ which is the finest partition of the closure of the support $\mathbf{U}(\mathcal{A})$ induced by the total intersections of the A_j with each other and with all their intersections recursively, where $1 \le k \le Q-1, N+1 \le Q \le 2N$, and ρ is constant over each $G_k \subseteq \mathbb{R}$. For full details see [3].

An example is shown on the left of Fig. 5. Assuming a common axis x shown for $x \in [0,4]$, the half-open intervals on the bottom are observed with the frequencies indicated by $m(A_j)$. The partition $\Gamma = \{G_k\}$ is shown above it with $\mathbf{U}(\Gamma) = [0,4] \supseteq \mathbf{U}(\rho)$. The resulting piecewise-constant plausibilistic trace $\rho(x)$ is shown on top, which is possibilistically subnormal since $\mathcal{A}$ is inconsistent.

4.2. Concatentation Lattice

Because of the linear ordering of $\mathbb{R}$, operations on the G_k portions of the domain of a random interval are very different from the operations on the focal sets of a finite random set.

Definition 4.4.1 (Concatenation) Assume two intervals $I_1, I_2 \in \mathcal{D}, I_1 := [l_1, r_1), I_2 := [l_2, r_2)$, and define the relation $I_1 < I_2 := r_1 \leq l_2$. Then the **concatenation** operation is

$$I_1 \uplus I_2 := \begin{cases} [l_1, r_2), & I_1 < I_2 \\ [l_2, r_1), & I_1 > I_2 \\ I_1 \cup I_2, & \text{otherwise} \end{cases} .$$

Note that $I_1 \cup I_2 \subseteq I_1 \uplus I_2$.

Now given a random interval $\mathcal{A}$, without loss of generality let the G_k be ordered so that $G_k < G_{k+1}, 1 \leq k \leq Q-2$. Then Γ is the set of atoms of a lattice $\mathcal{L} \subseteq 2^\Gamma$ with operations

$$\forall I_1, I_2 \in \mathcal{L}, \qquad I_1 \wedge I_2 = I_1 \cap I_2, \quad I_1 \vee I_2 = I_1 \uplus I_2.$$

An element $I \in \mathcal{L}$ maps to a unique set of atoms denoted $\{G_{k_I}\} \subseteq \Gamma$ for $I_l \leq k_I \leq I_u$ such that $I = \biguplus_{k=I_l}^{I_u} G_k$. Therefore denote $I_1 I_2 := I_1 \uplus I_2$, so that $I = G_{I_l} G_{I_l+1} \cdots G_{I_u-1} G_{I_u}$. $\mathcal{L}$ is not a sub-lattice of 2^Γ, is not closed under unions, and is not distributive or complemented.

4.3. Focused Consistent Transformations in $\mathcal{L}$

Since $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{L}$, therefore m can now be modified to be $m\colon \mathcal{L} \mapsto [0,1]$ with $m(\emptyset) = 0$ and $\sum_{I \in \mathcal{L}} m(I) = 1$ as before. In this space focused consistent tranformations now require an atomic normalizing interval $G_\kappa \in \Gamma$. The focused consistent transformation algorithm is then modified to create a new consistent random interval $\hat{\mathcal{A}}_\kappa$ by affecting the transforms

$$\forall A_j \in \mathcal{F}(\mathcal{A}), \quad A_j \mapsto \hat{A}_j := A_j \uplus G_\kappa.$$

We have the following:

Proposition 4.4.2 Given an inconsistent random interval $\mathcal{A}$ with focus G_κ, then $\forall I \in \mathcal{L}$,

$$\hat{m}^\kappa(I) = \begin{cases} \sum_{k=I_l}^{I_u} m(G_{I_l} G_{I_l+1} \cdots G_k), & I_u = \kappa, \\ \sum_{k=I_l}^{I_u} m(G_k G_{k+1} \cdots G_{I_u}), & I_l = \kappa, \\ m(I), & I_l < \kappa < I_u, \\ 0, & \text{otherwise} \end{cases} .$$

The right side of Fig. 5 shows the transformation of the left side when $\kappa = 4$. Note that not only is π possibilistically normal, but that some of the endpoints have been lost in the creation of the new partition $\hat{\Gamma}$. In particular $\hat{G}_2 = G_2 \uplus G_3 = G_2 \cup G_3$. Further, all the evidence supporting G_5 has been added to that of G_4.

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